"Alexandru Ioan Cuza" University of Iaşi

Mihai ANASTASIEI

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PREFACE

BY

Mihai ANASTASIEI

Retiring from the University is an opportunity for retrospection of one's life achievements wether it is teaching or wether it is research. Even though teaching may one day cease, the research activity may go on indefinitely. Therefore, retiring is not an ending. It is merely a step in one's career and perhaps a new beginning.

Looking back to forty years of writing and publishing scientific papers, I decided to present to the scientific community a selection of my scientific works. I chose mostly articles published in prestigious journals or Proceedings that made a certain impact in the scientific world. I have selected thirty two out of some ninety papers, on the following subjects :

1. Geometry of conformal and spin structures on Hilbert manifolds,
2. Geometry of tangent bundle of a Finsler or a Lagrange manifold,
3. Applications of techniques from Finsler geometry to Mechanics and Physics,
4. Geometry of total space of a vector bundle,
5. Applications of Lie algebroids to Mechanics.

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CONFORMAL STRUCTURES ON BANACH VECTOR BUNDLES

BY

M. ANASTASIEI

In this paper, conformal structures, in particular Weyl structures on Banach vector bundles are defined. We prove that there is a one-to-one correspondence between the set of conformal structures on a Banach vector bundle and the set of reductions of its structural group to the conformal group.

The existence and the uniqueness of a connection without torsion, compatible with a Weyl structure on a Banach manifold are proved.

1 Linear conformal space. Conformal group

Let $\mathbf{E}$ be a real, infinite-dimensional linear space.

Definition 1.1. A conformal structure on $\mathbf{E}$ is a set $C(\mathbf{E})$ of scalar products on $\mathbf{E}$, denoted by $(\ ,\ )_a$, $a \in \mathcal{I}$ which satisfy

$$(\ ,\ )_a = \lambda_{ab}(\ ,\ )_b \quad a, b \in \mathcal{I} \tag{1.1}$$

where λ_{ab} is a positive real number, and $\mathcal{I}$ a set of indices.

Definition 1.2. The linear space $\mathbf{E}$ with conformal structure $C(\mathbf{E})$ is called a *conformal space.*

Remark 1.1. In the conformal space $\mathbf{E}$ the angle between two vectors can be defined by

$$\cos(u, v) = \frac{(u, v)_a}{\|u\|_a \|v\|_a}, \ \forall u, v \in \mathbf{E},\ a \in \mathcal{I}, \tag{1.2}$$

where $\|u\|_a = \sqrt{(u, u)_a}$, the ratio of their lengths is well defined but their absolute lengths are not defined.

If $(\ ,\ )$ is a fixed element of $C(\mathbf{E})$, (1.1) can be replaced by

$$(\ ,\ )_a = \lambda_a(\ ,\ ) \quad \forall a \in \mathcal{I}. \tag{1.3}$$

This implies that all norms $u \to \|u\|_a$ are equivalent to the fixed norm $u \to \|u\| = \sqrt{(u, u)}$. In the following, the space $\mathbf{E}$ with the norm $\|\cdot\|$ will be supposed to be complete.

Remark 1.2. Let $R_i(\mathbf{E})$ be the set of all scalar products on $\mathbf{E}$. We have

$$C(\mathbf{E}) \subseteq R_i(\mathbf{E}) \subseteq L_s^2(\mathbf{E}). \tag{1.4}$$

(Here $L_s^2(\mathbf{E})$ is the linear space of bilinear and symmetric maps $s : \mathbf{E} \times \mathbf{E} \to \mathbf{E}$.)

Let $L(\mathbf{E})$ be the linear space of linear bounded operators on $\mathbf{E}$.

Definition 1.3. We say that $A \in L(\mathbf{E})$ preserves the conformal structure of $\mathbf{E}$ if there is a unique $a \in \mathcal{J}$ such that

$$(Au, Av) = (u, v)_a, \quad u, v \in \mathbf{E}.$$

If A^* denotes the adjoint operator of A with respect to the scalar product $(\ ,\ )$, then we have

Theorem 1.1. *Let $\mathbf{E}$ be a conformed space and $A \in L(\mathbf{E})$. The following conditions are equivalent*

1) A preserves the conformal structure of $\mathbf{E}$;

2) There is a unique real number $k_A > 0$ such that

$$A^*A = k_A I \quad (I \text{ identity operator});$$

3) A preserves the angle between vectors of $\mathbf{E}$.

Proof. Obvious.

Definition 1.4. An operator $A \in L(\mathbf{E})$ which satisfies one of the conditions of Theorem 1.1 will be called a *conformal operator*.

Let $CO(\mathbf{E})$ be the set of invertible conformal operators and $O(\mathbf{E})$ the set of invertible operators which satisfy $A^*A = I$. We immediately obtain the following

Theorem 1.2. *The sets $CO(\mathbf{E})$ and $O(\mathbf{E})$ are subgroups of the group $GL(\mathbf{E})$ of all invertible operators.*

We call $CO(\mathbf{E})$ and $O(\mathbf{E})$ the conformal group and the orthogonal group of $\mathbf{E}$, respectively.

Theorem 1.3. *There is an isomorphism*

$$\alpha : CO(\mathbf{E}) \to O(\mathbf{E}) \times R_+^*,$$

where R_+^* is the positive real multiplicative group.

Proof. For $A \in CO(\mathbf{E})$ and $A^*A = k_A I$, we put $\alpha(A) = \left(\frac{1}{\sqrt{k_A}} A, k_A\right)$ and for $(B, 1) \in O(\mathbf{E}) \times R_+^*$, $\alpha^{-1}(B, 1) = \sqrt{1} B$.

Let $GL(\mathbf{E})$ be endowed with the topology induced by the norm topology of $L(\mathbf{E})$. We identify the group R_+^* with the homotheties group of $\mathbf{E}$. The following result is obvious.

Theorem 1.4. *The subgroups $CO(\mathbf{E})$, $O(\mathbf{E})$ and R_+^* are closed, and the map α from Theorem 1.3 is a topological isomorphism.*

2 Conformal structures on a Banach vector bundle

All vector bundles, manifolds and maps considered in the following sections will be assumed of class C^∞.

Let E and M be manifolds, modeled on Banach spaces, and suppose M connected. Let $\pi : E \to M$ be a vector bundle with fibre the conformal space $\mathbf{E}$. We denote by $Ri(\pi)$ the set of Riemannian metrics on π, and we define the following equivalence relation:

$$\Lambda : g \sim g' \Leftrightarrow g' = e^\lambda \cdot g, \quad g, g' \in Ri(\pi),$$

where λ is a smooth function on M. (The use of exponential function is a handy way of ensuring positivity).

Definition 2.1. A conformal structure on π is an equivalence class $\mathbf{C}$ with respect to Λ of Riemannian metrics on π.

Remark 2.1. If the equivalence class $\mathbf{C}$ contains only one element, we obtain a Riemannian structure on π.

The proofs of the two following theorems are standard. (See [3, Ch. 7] for the particular case of the Riemannian structure).

Theorem 2.1. *Let $\pi : E \to M$ be a vector bundle with fibre $\mathbf{E}$ and suppose M admits a partition of unity. Then the vector bundle π admits a conformal structure.*

Theorem 2.2. *Let $\pi : E \to M$ and $\pi' : E' \to M'$ be vector bundles and let $f : E' \to E$ be a bundle morphism such that the map $f_{p'} : E'_{p'} \to E_{f(p')}$, where $E'_{p'} = \pi'^{-1}(p')$ and $E_{f(p')} = \pi^{-1}(f(p'))$, is injective and such that $f_{p'}(E'_{p'})$ has a complementary closed subspace in $E_{f(p')}$. Then a conformal structure on π canonically induces a conformal structure on π'.*

Definition 2.2. The vector bundle $\pi : E \to M$ with fibre $\mathbf{E}$ admits a reduction of its structural group to $CO(\mathbf{E})$, if and only if there exists a bundle atlas $(U_i, \tau_i)_{i\in I}$, such that the maps $(\tau_j \circ \tau_i^{-1})_p : \mathbf{E} \to \mathbf{E}$ for each p of $U_i \cap U_j$ belong to the group $CO(\mathbf{E})$.

Theorem 2.3. *Let $\pi : E \to M$ be a vector bundle with fibre $\mathbf{E}$. There exists a one-to-one correspondence between the set of reductions of the structural group to the conformal group and the set of conformal structures on π.*

Proof. Every reduction of π to the conformal group $CO(\mathbf{E})$ determines the conformal structure of π. Indeed, we define

$$g_{a,p}(v, w) = (\tau_{i,p}, v, \tau_{i,p}, w)_a, \ \forall v, w \in \mathbf{E} \text{ and } a \in \mathcal{J}.$$

The maps $g_a : p \to g_{a,p}$ define the sections of vector bundle $L_s^2(\pi)$ (see [3, Ch. 3] for the definition of this vector bundle) and the set $\{g_a\}$ is a conformal structure on π.

Conversely, let $\{g_j\}_{j\in\mathbf{J}}$ be a conformal structure on π and let $(U_i, \tau_i)_{i\in\mathbf{I}}$ be a bundle atlas for π. We consider an arbitrary map $\varepsilon : \mathbf{I} \to \mathbf{J}$ and let $g_i^{\varepsilon(i)}$

be the induced metric by g_j with $j = \varepsilon(i)$, on $U_i \times \mathbf{E}$ by the isomorphism τ_i. There exists a positive definite symmetric operator $A^{\varepsilon(i)}_{i,p}$ such that

$$g^{\varepsilon(i)}_{i,p}(v, w) = (A^{\varepsilon(i)}_{i,p} v, w), \ \forall p \in U_i, \ v, w \in \mathbf{E}.$$

We denote $B_{i,p} = \sqrt{A^{\varepsilon(i)}_{i,p}}$ and we put $\sigma_i = B_{i,p} \circ \tau_{i,p}$. Then (U_i, σ_i) is the bundle atlas we looked for. It is sufficient to prove that $B_i : U_i \times \mathbf{E} \to U_i \times \mathbf{E}$ which is defined on fibres by $B_{i,p}$ map $g^{\varepsilon(i)}_i$ on the scalar product $(\ , \ )$ of $\mathbf{E}$. But we have

$$(B_{ip} v, B_{i,p} w) = (A^{\varepsilon(i)}_{i,p} v, w) = g^{\varepsilon(i)}_i (v, w),$$

since $B_{i,p}$ is symmetric.

Definition 2.3. Let $\pi : E \to M$ be a vector bundle with a conformal structure $\mathbb{C}$; $\mathbb{C}$ is called a Weyl structure if and only if there exists a map $W : \mathbb{C} \to C^\infty(T^*M)$ which satisfies

$$W(e^\lambda \cdot g) = W(g) - d\lambda,$$

where $C^\infty(T^*M)$ denotes the set of sections in cotangent bundle of M.

Remark 2.1. A Riemannian metric g and a $1-$form η on M determines a Weyl structure, namely $W : \mathbb{C} \to C^\infty(T^*M)$, where $\mathbb{C}$ is the equivalence class of g and $W(e^\lambda g) = \eta - d\lambda$.

Theorem 2.4. *Let $\pi : E \to M$ and $\pi' : E' \to M'$ be vector bundles with conformal structures and $f : E' \to E$ a bundle morphism compatible with this conformal structure (in the sense of Theorem 2.2). Every Weyl structure on π canonically induces a Weyl structure on π'.*

Proof. If (g, η) defines Weyl structure on π, then $(f^*g, f^*\eta)$ defines a Weyl structure on π'.

Theorem 2.5. *Let $\pi : E \to M$ be a vector bundle with a conformal structure, where M admits a partition of unity. Then π admits a Weyl structure.*

Proof. It is sufficient to prove that there is a global 1-form on M. But this follows from [1, Lemma 1.3].

3 Connections compatibles with conformal structures

We shall give below some results from the connections theory on Banach vector bundles which we shall use later.

Theorem 3.1. *Let $\pi : E \to M$ be a vector bundle and M admits a partition of unity; then*

i) there exists a connection map K for π,

ii) there exists a canonic bijective map from the set of connection maps on π to the set of covariant derivatives on π given by

$$K \circ T\xi = \nabla\xi, \quad \forall \xi \in \mathcal{X}_E(M), \tag{3.1}$$

where $\mathcal{X}_E(M)$ is the set of the sections in π.

The proof is given in [1, Theorem 2.2].

Remark 3.1. a) $\nabla\xi$ is considered as a section in $L(\tau,\pi) : L(TM,E) \to M$ where $\tau : TM \to M$ is tangent bundle.

b) The implication $K \to \nabla$ is given by (3.1) and without the hypothesis of existence of the partition of unity on M.

Let $c : [0,1] \to M$ be a piecewise differentiable curve on M. We denote by $P_c|_{[t,t_0]}$, where $t, t_0 \in (0,1)$, the parallel displacement from $E_{c(t)}$ to $E_{c(t_0)}$ defined by the connection ∇, K. The map $\widetilde{Q}_c : c^*E \to (0,1) \times E_{c(t_0)}$ defined by

$$\widetilde{Q}_c(t,v) = (t, P_c|_{[t,t_0]}v) \quad (t,v) \in c^*E. \tag{3.2}$$

is a vector bundles isomorphism. See [1, Theorem 3.5].

Let $\mathcal{X}_E(c)$ be the vector space of section in π along the curve c and let $C^\infty((0,1), E_{c(t_0)})$ be the vector space of maps of class C^∞ from $(0,1)$, to $E_{c(t_0)}$. We consider the map $Q_c : \mathcal{X}_E(c) \to C^\infty((0,1), E_{c(t_0)})$ defined by

$$Y \to Q_cY = pr_2 \circ \widetilde{Q}_c(t, Y(t)) \quad \forall Y \in \mathcal{X}_E(c),\ t \in (0,1). \tag{3.3}$$

Theorem 3.2. *a) Q_c is a vector space isomorphism;*

b) $\dfrac{d}{dt}(Q_cY) = Q_c(\nabla_c, Y)$ where ∇_cY is covariant differentiation of section Y along curve c.

For proof see [1, Theorem 3.6].

In [4], the holonomy group of connection ∇, K with reference point p, denoted by $\Phi(p)$, is defined. For each p of M, the group $\Phi(p)$ can be realized as a subgroup of the structural group of π. The Theorem 2.11 of [4] suggests the following

Definition 3.1. Let $\pi : E \to M$ be a vector bundle with a conformal structure $\mathbb{C}$. The connection ∇, K is compatible with the conformal structure $\mathbb{C}$ if and only if

$$\Phi(p) \subseteq CO(\mathbf{E}), \quad \forall p \in M, \tag{3.4}$$

where $\mathbf{E}$ is the fibre of π.

In the interesting case when $\mathbb{C}$ is in the same time a Weyl structure, we will use the following

Definition 3.2. (See [2]). Let $\pi : E \to M$ be a vector bundle with a Weyl structure (g,η). The connection ∇, K is compatible with the Weyl structure (g,η) if and only if along every curve $c : [0,1] \to M$ and for at least one g from $\mathbb{C}$,

$$g_p(Q^t_CY, Q^t_CZ) = \exp\left[\int_0^t c^*\eta\right] g_{c(t)}(Y_t, Z_t), \tag{3.5}$$

where $Q^t_c = P_C|_{[t,t_0]}, p = c(0)$ and $Y_t, Z_t \in E_{c(t)}$.

Remark 3.2. If condition (3.5) is satisfied by one $g \in \mathbb{C}$, it will be satisfied by each $g' = e^{\lambda} \cdot g$ from $\mathbb{C}$.

Theorem 3.3. *Let $\pi : E \to M$ be a vector bundle with the Weyl structure (g, η) and let ∇, K be a connection on π.*

The following assertions are equivalent:

1) *The connection ∇, K is compatible with the Weyl structure (g, η);*

2) $\dfrac{d}{dt} g_c(Y, Z) = g_c(\nabla_c Y, Z) + g_c(Y, \nabla_c Z) - c^*\eta \cdot g_c(Y, Z)$, $\forall c : [0, 1] \to M$ *and* $Y, Z \in \mathcal{X}_E(M)$;

3) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - \eta(X) g(Y, Z)$ $\forall X \in \mathcal{X}_{TM}(M)$ *and* $Y, Z \in \mathcal{X}_E(M)$

Proof. 1) $\to$ 2). For each curve c with $c(0) = p$, we shall denote $Y_{c(t)} = Y_t$, $Z_{c(t)} = Z_t$, $Y_p = Y$, $Z_p = Z$ for $Y, Z \in \mathcal{X}_E(M)$. It follows from (3.4) and b) of Theorem 3.2 that

$$\frac{d}{dt} g_c(Y, Z) = \lim_{t \to 0} \frac{1}{t} (g_{c(t)}(Y_t, Z_t) - g_p(Y, Z)) =$$

$$= \lim_{t \to 0} \frac{1}{t} \left(\exp\left[-\int_0^t c^*\eta \right] g_p(Q_c^t Y_t, Q_c^t Z_t) - g_p(Y, Z) \right) =$$

$$= \lim_{t \to 0} \frac{1}{t} \exp\left[-\int_0^t c^*\eta \right] (g_p(Q_c^t Y_t, Q_c^t Z_t) - g_p(Y, Z)) +$$

$$+ g_p(Y, Z) \lim_{t \to 0} \frac{1}{t} \left(\exp\left[-\int_0^t c^*\eta \right]_{-1} = g_p(\lim_{t \to 0} \frac{1}{t} (Q_c Y_t - Y), Z) + \right.$$

$$+ g_p(Y, \lim_{t \to 0} \frac{1}{t} (Q_c^t Z_t - Z)) + g_p(Y, Z) \frac{d}{dt} \exp\left[-\int_0^t c^*\eta \right] =$$

$$= g_p(\nabla_c Y, Z) + g_p(Y, \nabla_c Z) - c^*\eta \cdot g_p(Y, Z) \text{ i.e.2)}.$$

2) $\to$ 1). Let Y, Z be parallel sections in π along c, i.e $Q_c^t Y_t = Y_p$ and $Q_c^t Z_t = Z_p$. Assertion 2) of Theorem becomes

$$\frac{d}{dt} g_{c(t)}(Y_t, Z_t) = -c^*\eta \cdot g_{c(t)}(Y_t, Z_t), \tag{3.6}$$

and we get (3.5) by integration.

The proof of 2) $\to$ 3) can be obtained in the same way as in the Riemannian case. See [1, Theorem 3.8].

Definition 3.3. We shall say that a manifold M modeled by the conformal space $\mathbf{M}$ is endowed with a conformal structure if there is a collection of charts (U_i, φ_i), covering M and satisfying

$$D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(p)} \in CO(\mathbf{M}) \text{ for all } i, j \text{ and } p \in U_i \cap U_j, \tag{3.7}$$

where D denotes the differentiation operator.

Theorem 3.4. *A manifold M modeled by a conformal space $\mathbf{M}$ admits a conformal structure if and only if the tangent bundle TM admits a conformal structure.*

Proof. Let (U_i, φ_i) be the collection of charts which defines the conformal structure on M. The transition maps of TM are $D(\varphi_j \circ \varphi^{-1})_{\varphi_i(p)}$ and belong to $CO(\mathbf{M})$ i.e. TM admits a reduction to conformal group, therefore a conformal structure by Theorem 2.3.

Conversely, a conformal structure on TM induces a reduction of this vector bundle to the conformal group $CO(\mathbf{M})$ i.e. the maps $D(\varphi_j \circ \varphi_i^{-1})_{\varphi_i(p)}$ belong to $CO(\mathbf{M})$.

Definition 3.4. A conformal manifold M is called a Weyl manifold if and only if the conformal structure of TM is a Weyl structure.

Theorem 3.5. *Let M be a Weyl manifold, modeled by the conformal space $\mathbf{M}$. There exists a unique connection ∇, K, such that*
i) $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - \eta(X)g(Y,Z)$ for $X, Y, Z \in \mathcal{X}_{TM}(M)$
ii) $T(X,Y) \overset{def}{=} \nabla_X Y - \nabla_Y X - [X,Y] = 0$, $\forall X, Y \in \mathcal{X}_{TM}(M)$ where (g,η) is the Weyl structure of TM.

Proof. Existence. Let (U,φ) be a chart for M at p. We consider the following equation with Fréchet derivatives

$$\begin{aligned}2g_\varphi(\Gamma_{\varphi(p)}((u,v),w)) &= Dg_\varphi|_{\varphi(p)}(u)(v,w)+ \\ +Dg_\varphi|_{\varphi(p)}(v)(u,w) - Dg_\varphi|_{\varphi(p)}(w)(u,v) &+ \eta_\varphi(u)g_\varphi(v,w)+ \\ +\eta_\varphi(v)g_\varphi(u,w) - \eta_\varphi(w)g_\varphi(u,v), &\quad \forall u,v,w \in \mathbf{M},\end{aligned} \tag{3.8}$$

where g_φ and η_φ are local representatives of g and η, respectively. This equation defines a map $\Gamma_{\varphi(p)} \in L^2_s(M,M)$, such that $\varphi(p) \to \Gamma_{\varphi(p)}$ is of class C^∞. As the $\Gamma_{\varphi(p)}$ satisfies the usual transformation formula of a local connector, under change of trivialization, it defines a connection on M. The connection such obtained satisfies i) and ii) of Theorem. Indeed, ii) has the following local expression

$$T(X,Y)_{\varphi(p)} = \Gamma_{\varphi(p)}(X_{\varphi(p)}, Y_{\varphi(p)}) - \Gamma_{\varphi(p)}(Y_{\varphi(p)}, X_{\varphi(p)}) = 0.$$

This equality is satisfied because $\Gamma_{\varphi(p)}$ is a bilinear symmetric map. The local expression of condition i) is

$$Dg_\varphi|_{\varphi(p)}(X_{\varphi(p)})(Y_{\varphi(p)}, Z_{\varphi(p)}) = g_{\varphi(p)}(\Gamma_{\varphi(p)}(X_{\varphi(p)}), Z_{\varphi(p)})+$$

$$g_{\varphi(p)}(Y_{\varphi(p)}, \Gamma_{\varphi(p)}(X_{\varphi(p)}, Z_{\varphi(p)})) - \eta_{\varphi(p)}(X_{\varphi(p)})g_{\varphi(p)}(Y_{\varphi(p)}, Z_{\varphi(p)}).$$

This equality can be easily verified using (3.8).

Uniqueness. Let $\Gamma'_{\varphi(p)}$ be another local connector which verifies i) and ii). It follows that $\Gamma'_{\varphi(p)}$ must satisfy equation (3.8) i.e, $\Gamma'_{\varphi(p)} = \Gamma_{\varphi(p)}$.

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GEOMETRIA DIFFERENZIALE
Affine transformations on Banach manifolds

Nota di M. ANASTASIEI
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Abstract

RIASSUNTO. In questo lavoro si dimostra che, in determinate condizioni, il gruppo delle trasformazioni affini di una varietà riemanniana di dimensione infinita coincide col gruppo delle isometrie. Un risultato di questo tipo, nel caso della dimensione finita, è stato precedentemente ottenuto da S. Kabayashi [2].

In this paper we prove that, under certain conditions, the group of affine transformations of a Riemannian infinite-dimensional manifold M is equal to the group of isometries of M. A result of the same type, in the finite-dimensional case, has been obtained by S. Kobayashi [2].

1 Affine morphisms of Banach manifolds

We work in the category of infinite-dimensional manifolds of class C^∞. Let M be a Banach manifold. We suppose the existence of a connection map $K : T^2M \to TM$ and denote by ∇ the covariant differentiation associated to it, see [1]. For X, Y in $\chi(M)$, the $F(M)-$module of vector fields on M,we set

$$\nabla_X Y = K \circ TY(X), \quad \frac{Dc}{dt} = K \circ T\dot{c}, \tag{1.1}$$

where TY is the tangent map of $Y : M \to TM$ and $c : [0,1] \to M$ is a curve on M. The holonomy groups, denoted by $\Phi(p)$, for p in M, were introduced and studied in [4].

Definition 1.1. A Banach manifold M, endowed with a connection map, is said to be irreducible if $\Phi(p)$ does not have any trivial invariant subspace. Otherwise, it is called reducible.

Definition 1.2. Let M and M' be endowed with the connection maps K and K', respectively. A morphism $f : M \to M'$ is called affine if, and only if,

$$Tf \circ K = K' \circ T^2 f. \tag{1.2}$$

If $M = M'$ and f is a diffeomorphism, we say that f is an affine transformation.

In the following theorem we collect some facts about affine morphisms, needed in the next section; for the proof see [5].

Theorem 1.1. *Let M and M' be Banach manifolds with the connection maps K and K', respectively. Suppose $f : M \to M'$ is an affine diffeomorphism. Then:*

a) $Tf \circ \tau_c = \tau'_{f\circ c} \circ Tf$ for every curve c, where τ_c (resp. $\tau'_{f\circ c}$) denotes the parallel displacement along the curve c (resp. $f \circ c$);

b) $Tf(\nabla_X Y) = \nabla'_{TfX} TfY$, for all X, Y in $\chi(M)$;

c) $Tf \circ R(X,Y)Z = R'(TfX, TfX)TfZ$, for all X, Y, Z in $\chi(M)$, where R (resp. R') denotes the curvature tensor field associated with K (resp. K').

Let (M, g) be a Riemannian manifold. As in the finite dimensional case, the sectional curvature for a $2-$plane $\sigma = \{X, Y\}$ in T_pM (the tangent space at p in M) is defined by

$$K_p(\sigma) = \frac{g(R(X,Y)Y,X)}{g(X,Y)g(Y,Y) - g^2(X,Y)}. \tag{1.3}$$

Definition 1.3. Let (M, g) and (M', g') be Riemannian manifolds. A morphism $f : M \to M$ is called a homothety if

$$g'(TfX, TfY) = c^2 g(X,Y) \text{ for any } X, Y \text{ in } \chi(M). \tag{1.4}$$

If in (1.4) $c = 1$, then f is an isometry.

It is proved in [1, p.38] that every isometry is an affine morphism (with respect to the unique connections without torsion defined by g and g', respectively).

In particular, the group of isometries of M is a subgroup of the group of affine transformations of M.

2 The main results

The purpose of this section is to prove Theorems 2.1 and 2.2.

Theorem 2.1. *Let (M, g) be an irreducible Riemannian manifold, with bounded and non-identically zero sectional curvature. Then, the group of affine transformations of (M, g) is equal to the group of isometries of (M, g).*

Proof. The proof will be given in three steps.

Step 1. Every homothety is an affine transformation. Using a homothety f, we define a new Riemannian metric on M by $g'(X, Y) = g(TfX, TfY) = c^2 g(X, Y)$. Obviously, $f : (M, g') \to (M, g)$ is an isometry, hence an affine transformation. But, from the definitions of the Riemannian connection [1, p. 36], it follows that the connection defined by g' and g coincide; therefore $f : (M, g) \to (M, g)$ is an affine transformation.

Step 2. If (M, g) is irreducible, every affine transformation is a homothety. For this we need the following

Lemma. *Let H be a real Hilbert space, $O(H)$ the orthogonal group and S a subgroup of $O(H)$ which acts irreducibly on H. If g is a symmetric and*

bilinear form on H, invariant under the action of S, then there is a constant c such that $g(u,v) = c(u,v)$ for all u,v in H, $(\ ,\ )$ being the standard inner product of H.

Proof of Lemma. There exists a symmetric operator A such that $g(u,v) = (Au,v)$. Let s be an element of S. From $g(su,sv) = g(u,v)$ (invariance of g) it follows $As = sA$ for all s in S and from Theorem 6, Appendix II of [3], it follows that there exists a constant c, such that $A = cI$ (where I is the identity operator) and therefore $g(u,v) = c(u,v)$. We remark that, if g is positive definite, the constant c must be positive.

We give now the proof of *Step 2*.

For p in M there are two inner products g_p and g'_p on T_pM, where $g'_p(X,Y) = g(T_pfX, T_pfY)$. As f is an affine transformation g is invariant under the action of $\Phi(p)$ which is a subgroup of the orthogonal group $O(T_pM)$ (with respect to the inner product g). We are in position to apply the Lemma and we obtain $g'_p = c_p^2 g_p$. But g and g' are the parallel tensor fields with respect to the Riemannian connection defined by g, therefore c_p does not depend on p i.e. f is a homothety.

Step 3. In the hypothesis of Theorem 2.1, every affine transformation is an isometry. Let f be an affine transformation of M. By Step 2, f is a homothety. If $c = 1$, the proof is complete. Suppose $c < 1$, otherwise we may use f^{-1} and denote by $K < +\infty$ the bound of the sectional curvature. For every p in M and the $2-$plane $\sigma = \{X,Y\}$ in T_pM we have

$$|K(\sigma)| = c^{2m}|K_{f^m(p)}((T_pf)^m X,\ (T_pf)^m Y| \leq c^{2m}\cdot K,$$

and, for $m \to \infty$, we obtain $K_p(X,Y) \equiv 0$ which is a contradiction.

In the case of M irreducible and complete, the hypothesis "bounded sectional curvature" can be *weakened*. Firstly, we prove

Lemma 2.1. *Let (M,g) be a complete Riemannian manifold. Every strict homothety (i.e. with $c \neq 1$) of M, has a fixed point.*

Proof. (M,g) is a complete metric space with respect to the metric $d(p,q) = \inf\limits_b \left\{\int_0^1 g(\dot b,\dot b)^{\frac{1}{2}}dt\right\}$ for all curves b on M, with $b(0) = p$ and $b(1) = q$, see [5].

Let f be a homothety with $c < 1$, otherwise we may take f^{-1}. We have

$$d(f(p),f(q)) = \inf\left\{\int_0^1 g(\dot{\widehat{f\circ b}},\dot{\widehat{f\circ b}})^{\frac{1}{2}}dt\right\} \leq c\inf\left\{\int_0^1 g(\dot b,\dot b)^{\frac{1}{2}}dt\right\} \leq cd,$$

therefore f is a contraction map. It follows that f has a fixed point.

Now we give the following

Definition 2.1. The Riemannian manifold (M,g) is said to be with locally bounded sectional curvature if any p in M admits a closed neighborhood on which the sectional curvature is bounded.

Theorem 2.2. *Let (M,g) be a complete and irreducible Riemannian manifold with locally bounded and non-identically zero sectional curvature. Then, the group of affine transformations of M is equal to the group of isometries of M.*

Proof. By Step 2 of the proof of Theorem 2.1, every affine transformation f is a homothety and therefore by Lemma 2.1, has a fixed point, denoted by p_0. Let U be a closed neighborhood of p_0 on which the sectional curvature is bounded by $K < +\infty$. Suppose $c < 1$ and we have

$$d(p_0, f^m(p)) = d(f^m(p_0), f^m(p)) \leq c^m d(p_0, p),$$

for all p in M; hence there exists an $m_0 > m$ such that for $m_0 > m$, $f^m(p)$ belongs to U. From

$$|K_p(X, Y)| = c^{2m} |K_{f^m(p)}((T_p f)^m X, (T_p f)^m Y)| \leq c^{2m} \cdot K$$

it follows, when $m \to \infty$, $K_p(X, Y) \equiv 0$ which is a contradiction.

Remark 2 .1. The hypothesis of Theorem 2.1 are satisfied by a $\delta-$pinched Riemannian manifold (i.e. there exists a constant $0 < \delta < 1$ such that $\delta < K_p < 1$, for every p in M).

Remark 2.2. When M is a finite-dimensional Riemannian manifold, every p in M admits a neighborhood such that $\overline{U}$ is compact. As K_p is a continuous function, it follows that it is bounded; therefore M has locally bounded sectional curvature. In the case when M is complete and irreducible, we obtain the theorem by S. Kobayashi ([2]).

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ANALYSE MATHEMATIQUE

Structures spinorielles sur les variétés hilbertiennes

Note (*) de Mihai Anastasiei
présenté par M. André Lichnerowicz

Abstract

On donne quelques définitions et propriétés des structures spinorielles sur les variétés modelées par des espaces de Hilbert.

Some definitions and properties of the spin structures on the manifolds modeled by Hilbert spaces are given.

1 Introduction

Soit H un espace de Hilbert réel, séparable et de dimension infinie. Nous notons par $GL(H)$ le group général linéaire de H et par $O(H)$ le group orthogonal de H. Soit $P(H)$ un *classe de perturbation* pour l'anneau $L(H)$ des opérateurs linéaires bornés sur H et soit $GL_p(H) = \{X \in GL(H),\ X$ congruent à I modulo $P(H)\}$, ou I désigne l'opérateur identité sur H. Le sous-groupe $O(H)_p = O(H) \cap GL_p(H)$ a deux composantes connexes; soit $SO(H)_p$ sa composante connexe de l'identité. Si $P(H)$ coincide avec l'idéal des opérateurs nucléaires [resp. de Hilbert-Schmidt], les groupes $O(H)_p$, $SO(H)_p$ seront notés par $O(H)_1$, $SO(H)_1$ [resp. par $O(H)_2$, $SO(H)_2$].

Pierre de la Harpe a donné dans [3] une construction explicite du revêtement universel $\mathrm{Spin}(H)_1$ de $SO(H)_1$. Ultérieurement, R.J. Plymen et R.F. Streater ont donné dans [5] la construction explicite du revêtement universel $\mathrm{Spin}(H)_2$ de $SO(H)_2$. Les groupes $\mathrm{Spin}(H)_1$ at $\mathrm{Spin}(H)_2$ s'appellent les groupes spinoriels. Posons $\mathrm{Spin}(H) = \mathrm{Spin}(H)_1$ ou bien $\mathrm{Spin}(H)_2$ et $SO(H) = SO(H)_1$ ou bien $SO(H)_2$ et notons par $\rho : \mathrm{Spin}(H) \to SO(H)$, l'homomorphisme correspondant de revêtement.

Dans la suite, nous allons définir les structures spinorielles en utilisant les groupes $\mathrm{Spin}(H)$ et nous allons donner quelques propriétés de ces structures. Nous allons considérer aussi fibrés de Clifford.

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2 Définitions des structures spinorielles

Nous supposerons toujours la différentiabilité de classe C^∞. Soient M une variété différentiable, conexe, modelée sur un espace de Banach et soit $\pi : E \to M$ un fibré vectoriel (en abrégé f.v.) de fibre type H.

Définition 2.1. Nous appelons $P-$structure riemannienne sur le f.v. π, une réduction du groupe structural de π au groupe $O(H)_p$.

Remarquons que ces structures existent toujours, $GL(H)$ étant contractible (le théorème de Kuiper). Nous les étudierons dans un autre travail. Pour le groupe $SO(H)_1$ [resp. $SO(H)_2$] nous obtenons la structure riemannienne nucléaire orientée [resp. riemannienne de Hilbert–Schmidt orientée].

En supposant que le f.v. π a une réduction au groupe $SO(H)$ soit $P(M, SO(H))$ son fibré de repères [1], qui est un fibré principal (en abrégé f.p.) de base M et de groupe structural $SO(H)$.

Définition 2.2. Une structure spinorielle sur le f.v. π avec une réduction au groupe $SO(H)$ (ou sur le f.p. $P(M, SO(H))$) est une extension du f.p. $P(M, SO(H))$ associée à l'homomorphisme de revêtement ρ :Spin$(H) \to SO(H)$.

Nous notons par $\Sigma(M, \mathrm{Spin})$ une telle extension et par

$$\widetilde{\rho} : \Sigma(M, \mathrm{Spin}(H)) \to P(M, SO(H))$$

l'homomorphisme qui correspond à l'homomorphisme ρ.

Remarque 2.1. La définition 2.2 est équivalente à la définition donnée par A. Lichnerowicz [6].

Remarque 2.2. Comme le f. p. $P(M, SO(H))$ est déterminé, à un isomorphisme près, par un recouvrement ouvert $\{U_i\}$ et un cocycle $g_{ij} : U_i \cap U_j \to SO(H)$ (1), le f. p. $\sum(M, \mathrm{Spin}(H))$ (s'il existe) est déterminé par un cocycle $\widetilde{g}_{ij} : U_i \cap U_j \to \mathrm{Spin}(H)$ tel que $\rho(\widetilde{g}_{ij}) = g_{ij}$. Cette remarque, réunie avec la possibilité d'identifier la classe d'isomorphie du f. p. $P(M, SO(H))$ avec un élément de l'ensemble de cohomologie $H^1(M, SO(H))$ est très utile.

Théorème 2.1. *Le f. p. $P(M, SO(H))$ admet une structure spinorielle si et seulement s'il existe un élement non nul $\sigma \in H^1(P, Z_2)$ tel que σ restreint a chaque fibre soit non trivial.*

Esquisse de preuve. Nous considérons σ comme un homomorphisme $\sigma : H_1(P) \to Z_2$ et nous définissons un homomorphisme $\sigma \circ \varphi_1 : \pi_1(P) \to Z_2$ ou φ_1 est l'homomorphisme de Hurewicz. Donc, $\ker(\sigma \circ \varphi_1)$ est un sous-groupe d'ordre deux dans $\pi_1(P)$. Comme P est localement contractible, il existe un revêtement d'ordre deux Σ de P, qui est l'espace total d'une extension de $P(M, SO(H))$ associée à ρ. La necessité est immédiate.

La définition d'une structure spinorielle qui decoule du théorème 2.1 est très utile pour les démonstrations des théorèmes suivantes [voir [7] pour la dimension finie ou pour le cas topologique].

Théorème 2.2. *L'ensemble des structures spinorielle, s'il n'est pas vide, est en bijection modulo l'isomorphisme de fibrés principaux avec $H^1(M, Z_2)$.*

Théorème 2.3. *Soient les f.v. π_1, et π_2, avec l'espace de base M et $\pi_1 \oplus \pi_2$, leur somme de Whitney. Si deux de ces fibres ont des structures spinorielles, le troisième est muni aussi d'une structure spinorielle.*

La suite exacte

$$1 \to Z_2 \to \mathrm{Spin}(H) \to SO(H) \to 1$$

induit, une suite exacte de groupes et d'ensemble de cohomologie [4]:

$$H^1(M, Z_2) \longrightarrow H^1(M, \mathrm{Spin}(H)) \longrightarrow H^1(M, SO(H)) \underset{v}{\longrightarrow} H^2(M, Z_2).$$

Il existe une structure spinorielle sur $P(M, SO(H))$ si et seutlement si $v(P) = 0$. Dans le cas $SO(H) = SO(H)_1$, $v(P) = w_2(P)$, [3]*, la deuxième classe de Stiefel-Whitney de $P(M, SO(H)_1)$.*

Soit M' une autre variété et soit $f : M' \to M$ un morphisme. Nous notons par Pf le f. p. sur M' induit par $P(M, SO(H))$ et f. *Si le f. p. $P(M, SO(H))$ admet une structure spinorielle, alors Pf admet une structure spinorielle.* Une propriété réciproque est donnée par le

Théorème 2.4. *Soit $f : M' \to M$ un f. p. de groupe structural G. Alors, G opère naturellement sur Pf et soit $u' \cdot G$ l'orbite de $u' \in Pf$. Si le f. p. Pf admet une structure spinorielle $\Sigma(M', \mathrm{Spin}(H))$ et si le groupe G opère sur Σ tel que la projection $\Sigma \to \Sigma/G$ est un f. p. de groupe structural G et $\widetilde{\rho}(u \cdot G) = \widetilde{\rho}(u) \cdot G$, où $u \cdot G$ est l'orbite de $u \in \Sigma$, alors $P(M, SO(H))$ admet une structure spinorielle.*

Démonstration. La variété Σ/G est muni d'une structure naturelle de f.p. de base M et de groupe structural $\mathrm{Spin}(H)$ par l'action $(u \cdot G)a = ua \cdot G$, pour $a \in$Spin(H), et projection $u \cdot G \to f(\alpha(u))$ avec α la projection $\Sigma \to M'$. L'homomorphisme canonique $\Sigma/G \to P$ est de la forme $u \cdot G \to f(f^*(u))$, où $f^* : Pf \to P$ est l'homomorphisme induit par f.

Dans le cas $G = Z_2$, nous obtenons un résultat qui a été démontré par I. Popovici [9] pour la dimension finie et le cas non orientable. Une structure spinorielle sur M sera par définition une structure spinorielle sur le fibré tangent TM (en supposant que M est modélée sur l'espace de Hilbert H).

Exemples. (a) Le f. p. trivial $M \times SO(H)$ admet toujours une structure spinorielle, unique si M est simplement connexe.

(b) Soit l'espace de Hilbert:

$$1_2 = \{x = (x_1, x_2, ...) \mid x_i \in \mathbb{R}, \sum_{i=1}^{\infty} x_i^2 < \infty\},$$

avec la base $c_1, c_2, ...$ Le tore hilbertien $T = 1_2 / \sum_{i=1}^{\infty} \mathbb{Z} c_1$ est un groupe de Lie-Hilbert. Chaque structure riemannienne nucléaire sur T est orientable car $w_1(T) = 0$ et il existe une structure spinorielle [rélatif à $\mathrm{Spin}(1_2)_1$] sur T, car $w_2(T) = 0$ [5].

(c) Des exemples plus sofistiqués découlent de [5].

3 Fibrés de Clifford

Maintenant, nous allons limiter nos considérations au groupe $SO(H)_1$. Soit $C(H)_1$ l'algèbre de Clifford avec sa structure de C^*-algèbre [3]. Chaque $*-$automorphisme ψ de $C(H)_1$ pour lequel $\psi(H) = H$, s'appelle de Bogoliubov. Le groupe de ces automorphismes est noté par $\text{Bog}(C(H)_1)$. Il existe un isomorphisme $C : O(H) \to \text{Bog}(C(H)_1)$ qui est aussi un isomorphisme de groupes de Lie-Banach.

Soit un f.p. $P(M, O(H))$. Son fibré associé avec la fibre $C(H)_1$ est un fibré en algèbres localement trivial, nommé fibré de Clifford. Nous notons par $\text{Bog}(C(H)_1)_i$ les automorphismes de Bogoliubov intérieurs et par $\text{Bog}(C(H)_1)_{ip}$ les automorphismes de Bogoliubov intérieures et paires.

L'isomorphisme $O(H)_1 \simeq \text{Bog}(C(H)_1)_i$ implique le:

Théorème 3.1. *Il existe une structure riemannienne nucléaire sur f.v. riemannienne si et seulement si le fibré de Clifford admet une réduction au groupe* $\text{Bog}(C(H)_1)_i$.

L'isomorphisme $SO(H)_1 \simeq \text{Bog}(C(H)_1)_{ip}$ [3] implique le

Théorème 3.2. *Une structure riemannienne nucléaire sur un f.v. riemannienne est orientable si et seulement si le fibré de Clifford admet une réduction au groupe* $\text{Bog}(C(H)_1)_{ip}$.

Dans le contexte des structures spinorielles, on peut étudier les champs spinoriels et les connexions spinorielles en dimension infinie. Ces aspects seront abordés dans un autre travail.

I. Pop et I. Popovici ont très utilement discuté sur ce travail.

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RIEMANNIAN P-STRUCTURES ON VECTOR BUNDLE

BY

M. ANASTASIEI

Introduction

Let H be a separable, real Hilbert space. Denote by $L(H)$ the algebra of bounded linear operators on H and by $\phi(H)$ the set of Fredholm operators on H. A two-sided ideal $P(H)$ of $L(H)$ is said to be a ϕ-perturbation class if $\phi(H) + P(H) = \phi(H)$ and $F(H) \subset P(H)$, where $F(H)$ is the two-sided ideal of the finite rank operators on H. Let $GL_P(H)$ be the group of those invertible operators on H, which can be written as $I + X$ with X in $P(H)$, where I is the identity operator. Denote by $O(H)$ the group of orthogonal operators on H and we put $O(H)_P = GL_P(H) \cap O(H)$. The group $O(H)_P$ has two connected components. Denote by $SO(H)_P$ the connected component of I.

Let M be a connected and paracompact manifold, locally diffeomorphic to a Banach space, and let be $\pi : E \to M$ a vector bundle having H as the type fibre. A $P-$structure on π is a reduction of its structural group to $GL_P(H)$. A vector bundle with a $P-$structure is called a $P-$bundle (see [3]). A reduction of the structural group of π to the group $O(H)_P$ will be called a Riemannian $P-$structure and a vector bundle with a Riemannian $P-$structure will be called a $PR-$bundle.

We are going to discuss the reduction of a $P-$bundle to a $PR-$bundle and to describe the morphisms of the $PR-$bundles.

1 On the reduction of a P-bundle to a PR-bundle

Let $\pi : E \to M$ be a vector bundle, as above. One say that π admits a reduction of its structural group to a subgroup G of $GL(H)$ if there exists a maximal collection of trivializations $(U_j, \phi_j)_{j\in J}$ with U_j open in M and

$$\phi_j : \pi^{-1}(U_j) \to U_j \times H$$

such that the maps

$$\phi_k \circ \phi_j^{-1} : U_j \cap U_k \to GL(H)$$

take their values in G.

Let g be a Riemannian metric on π. Using ϕ_j we can transport the restriction of the Riemannian metric g to $\pi^{-1}(U_j)$ on $U_j \times H$ and for a fixed point p in U_j we obtain a symmetric, bilinear and positive defined form on H whose corresponding operator (symmetric and positive) will be denoted by A_{jp}. The map $U_j \to L(H)$ given by $p \to A_{jp}$ is a morphism.

Definition 1.1. Let $\pi : E \to M$ be a vector bundle with a $P-$structure. A Riemannian metric g on π is said to be adapted to the $P-$structure of π if for every trivialization (U_j, ϕ_j) and for every p in U_j, there exists X_{jp} in $P(H)$ so that $A_{jp} = I + X_{jp}$.

Theorem 1.1. *Let $\pi : E \to M$ be a vector bundle with a $P-$structure. Then π admits a Riemannian $P-$structure if there exists a Riemannian metric g adapted to its $P-$structure.*

Proof. Let $(U_j, \phi_j)_{j\in J}$ be the maximal collection of trivialization of π, such that $\phi_k \circ \phi_j^{-1}$ are $GL(H)-$valued. Denote by g_j, the metric on $U_j \times H$ obtained from the restriction of g to $\pi^{-1}(U_j)$ and for every p in U_j we put $g_{jp}(v, w) = (A_{jp}v, w)$, where $(\ , \ )$ is the inner product on H. Let be A in $L(H)$; we agree to note by $\sqrt{A}$ an operator $B = \lim_n B_n$ where B_n is a sequence inductively defined by

$$B_{n+1} = \frac{1}{2}(B_n + B_n^{-1}A), \quad B_1 = I.$$

We define new trivializations for π by $\Psi_{jp} = B_{jp} \circ \phi_{jp}$, where $B_{jp} = \sqrt{A_{jp}}$ and $\phi_{jp} = \phi_j|_{\pi^{-1}(p)}$. Since for ever v and w in H, we have

$$(B_{jp}v, B_{jp}w) = (B_{jp}^2 v, w) = (A_{jp}v, w) = g_{jp}(v, w),$$

B_{jp} is an isometric map with respect to inner product on H and g_{jp}, hence $(\psi_k \circ \psi_j^{-1})\ (p) \in O(H)$. Since $A_{jp} = 1 + X_{jp}$ with X_{jp} in $P(H)$, it is easy to see that $\sqrt{A_{jp}} = I + Y_{jp}$ with Y_{jp} in $P(H)$.

Using this expression of $\sqrt{A_{jp}}$, we obtain:

$$(\psi_k \circ \psi_j^{-1})(p) = B_{kp} \circ \phi_{kp} \circ \phi_{jp}^{-1} \circ B_{jp}^{-1} = B_{kp}(I + Z_p)B_{jp}^{-1} =$$

$$= B_{kp} \circ B_{jp}^{-1} + B_{kp} \circ Z_p \circ B_{jp}^{-1} = (I + Y_{kp})(I + V_{jp}) + B_{kp} \circ Z_p \circ B_{jp}^{-1} =$$

$$= I + X_p$$

with X_p in $P(H)$, since $P(H)$ is a two-sided ideal. Therefore, $(U_j, \psi_j)_{i\in J}$ is the expected collection of trivializations of π.

Remark 1.1. By the Theorem 1.1, a $P-$structure on π and a Riemannian metric adapted to it, determine a Riemannian $P-$structure. Conversely, a Riemannian $P-$structure determines a $P-$structure (itself viewed as $P-$structure) and a Riemannian metric by $g_p(\xi, \eta) = (\phi_{jp}\xi, \phi_{jp}\eta)$ for p in M (the definition is correct because $\phi_{kp} \circ \phi_{jp}^{-1} \in O(H)$) whose associated operators A_{jp} are all equal to I.

Definition 1.2. A $PR-$bundle is orientable if it admits a reduction to the group $SO(H)_p$.

We have proved a criterion for the orientability of a $PR-$bundle in [l].

Consider for $F(H)$ the following $q-$norm ($1 \leq q \leq \infty$): $\|X\|_q =$ (trace $(\sqrt{X^*X})^q)^{1/q}$ for $1 \leq q < \infty$ and the usual norm for $q = \infty$. The closure of $F(H)$ in this $q-$norm will be denoted by $F_q(H)$. Each set $F_q(H)$ is a $\phi-$perturbation class and it corresponds to it a group denoted by $O(H)_q$. Some structures of great importance in the study of the spin structures on Hilbert manifolds are the reductions of the structural group of a vector bundle to $O(H)_1$, $O(H)_2$ respectively, named in [2], Riemannian nuclear structure, Riemannian Hilbert-Schmidt structure respectively. In the same Note there exists a condition for the reduction of a Riemannian vector bundle to a nuclear vector bundle; another criterion for the orientability of a Riemannian nuclear vector bundle is also given.

2 Morphisms of $PR-$bundles

Let $\phi_0(H)$ be the set of Fredholm operators of index 0.

Lemma 2.1. [4] *Every $T \in \phi_0(H)$ can be written as: $T = S + a$, where $S \in GL(H)$ and $a \in F(H)$.*

Definition 2.1. An operator $T \in \phi_0(H)$ is said to be an $O\phi_0-$operator if $a^*S + S^*a + a^*a = 0$, where S and a are as in Lemma 2.1.

Let π' be another vector bundle over M having the type fibre H. A morphism $f : \pi \to \pi'$ is called a ϕ_0-morphism if it is a ϕ_0-operator on each fibre.

Definition 2.2. Let π' be a Riemannian vector bundle. A morphism $f : \pi \to \pi'$ will be called an $O\phi_0-$morphism if for every trivialization (U, ϕ) and (V, ψ) with $f(U) \subset V$ of π and π' respectively, we have

$$(\psi \circ f \circ \phi^{-1})(x, v) = (f(x), f_1(x)v)$$

with $f_1(x)$ an $O\phi_0-$operator for every x in U.

Theorem 2.1. *Let π be a vector bundle and let π' be a $PR-$bundle. An $O\phi_0-$morphism $f : \pi \to \pi'$ induces a unique Riemannian $P-$structure on π, such that one has*

$$\psi \circ f \circ \phi^{-1}(x, v) = (f(x), v + a_0(x)v)$$

with $a_0(x)$ in $F(H)$ and $I + a_0(x)$ in $O(H)$, whenever (U, ϕ) and (V, ψ) with $f(U) \subset V$ are the trivializations of these Riemannian $P-$structures.

Proof. Let be (U, ϕ) a trivialization of π and (V_0, ψ_0) with $V_0 \subset f(U)$ a trivialization of π'. Therefore we have

$$(\psi \circ f \circ \phi^{-1})(x, v) = (f(x), f_1(x)v),$$

where $f_1(x)$ is an $O\phi_0-$operator for every x in U. By Lemma 2.1, $f_1(x) = S(x) + a(x)$, where $S(x) \in GL(H)$ and $a(x) \in F(H)$. Since $GL(H)$ is an open subset of $L(H)$, there exists a neighborhood U_0 of x, such that $S(U_0) \subset GL(H)$.

Define a new trivialization of π, $\phi_0 : \pi^{-1}(U_0) \to U_0 \times H$ by

$$\phi_0 \circ \phi^{-1}(x, v) = (f(x), S(x)v).$$

We have, using the definition of the $O\phi_0-$operators,

$$\psi_0 \circ f \circ \phi_0^{-1}(x, v) = \psi_0 \circ f \circ \phi^{-1} \circ \phi \circ \phi_0^{-1}(x, v) =$$

$$= \psi_0 \circ f \circ \phi^{-1}(x, S^{-1}(x)v) = (f(x), f_1(x)S^{-1}(x)v) =$$

$$= (f(x), (S(x) + a(x))S^{-1}(x)v) = (f(x), v + a(x)S^{-1}(x)v) =$$

$$= (f(x), v + a_0(x)v),$$

with $a_0(x)$ in $F(H)$ and $I + a_0(x)$ in $O(H)$.

Let (V_1, ψ_1) be another trivialization of π' and (U_1, ϕ_1) the trivialization of π associated to it by the above construction. Therefore we have

$$\psi_1 \circ f \circ \phi_1^{-1}(x, v) = (f(x), v + a_1(x)v)$$

with $a_1(x)$ in $F(H)$ and $I + a_1(x)$ in $O(H)$. We put $\psi_1 \circ \psi_0^{-1}(x, v) = (f(x), B'(x)v)$ with $B'(x)$ in $O(H)_p$ and $\phi_0 \circ \phi_1^{-1}(x, v) = (f(x), B(x)v)$. From $\psi_1 \circ f \circ \phi_1^{-1} = \psi_1 \circ \psi_0^{-1} \circ \psi_0 \circ f \circ \phi_0^{-1} \circ \phi \circ \phi_1^{-1}$ it follows

$$B'(x) \circ B(x)v + B'(x)a_0(x)B(x)v = v + a_1(x)v$$

hence, if we omit x,

$$(*) \qquad B'B + B'a_0B = I + a_1$$

or equivalently

$$(**) \qquad B'(I + a_0)B = I + a_1.$$

Therefore B is an orthogonal operator.

If we put $B' = I + b'$, from $(**)$ it follows

$$B = I + a_1 - b' \ B - a_0B - b'a_0B = I + a$$

with a in $F(H)$ since $F(H)$ is a two-sided ideal. Hence $(\phi_0 \circ \phi_1^{-1})(x) \in O(H)_p$ and the proof is complete.

Corollary 2.1. *Let $\pi : E \to M$ be a vector bundle with fibre H. An $O\phi_0-$morphism $f : E \to M \times H$ induces a unique Riemannian $P-$structure on E, so that for any trivialization (U, ϕ) of E, we have*

$$f \circ \phi^{-1}(x, v) = (f(x), v + a(x)v)$$

with $a(x)$ in $F(H)$, and $I + a(x)$ in $O(H)$.

Proof. One considers the trivial Riemannian $P-$structure on $M \times H$ and one applies the Theorem 2.1.

Corollary 2.2. *Let $f : \pi \to \pi'$ be an isomorphism of vector bundles. If π' admits an (orientable) Riemannian $P-$structure, there exists a unique (orientable) Riemannian $P-$structure on π such that for every trivialization (U, ϕ) and (V, ψ) with $f(U) \subset V$ of these Riemannian $P-$structures, we have*

$$(\psi \circ f \circ \phi^{-1})(x, v) = (x, v).$$

Proof. One repeats the proof of Theorem 2.1 with $a(x) = 0$ because f is an isomorphism. The relation (*) becomes $B'B = I$, hence $B' \in O(H)_p$ (resp. $SO(H)_p$) implies $B \in O(H)_p$ (resp. $SO(H)_p$).

Corollary 2.3. *Let be N and N' manifolds modeled by H. Suppose that N' admits an (orientable) Riemannian $P-$structure. A diffeomorphism $h : N \to N'$ induces an (orientable) Riemannian $P-$structure on N.*

Remark 2.1. Let $f : \pi \to \pi'$ an $O\phi_0-$morphism, where π' is a $PR-$bundle. Suppose that the Riemannian $P-$structure of π' is obtained from a $P-$structure and a Riemannian metric g. The Riemannian $P-$structure induced by f on π is not obtained from the $P-$structure induced by f and f^*g, since f^*g is not a Riemannian metric. However this happens in the context of the Corollary 2.2.

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SPIN STRUCTURES ON HILBERT MANIFOLDS

BY

M. ANASTASIEI

1 Introduction

Let H be a separable, real Hilbert space. We denote by $L(H)$ the algebra of bounded linear operators on H, by $O(H)$ the orthogonal operators on H and by I the identity operator on H. Let $GL_P(H)$ be the group of those invertible operators on H which can be written as $I+A$, where A is in a "perturbation class" $P(H)$ [2, p. 46] of $L(H)$. The group $O(H)_P = O(H) \cap GL_P(H)$ is doubly connected and we denote by $SO(H)_P$ the connected component of I.

Let $F(H)$ be the ideal of finite rank operators on H. We denote by $F(H)_p$ the closure of the ideal $F(H)$ in the $p-$norm defined by $\|X\|_p =(\text{trace}(X^*X)^{\frac{p}{2}})^{\frac{1}{p}}$ for $1 \leq p < \infty$ and by the usual norm, for $p = \infty$. The ideals $F(H)_p$ are "perturbation classes" for $L(H)$. For $p = 1$ (resp. $p = 2$) we obtain the ideal of nuclear operators (resp. the ideal of Hilbert-Schmid operators) and in this case the groups $O(H)_P$, $SO(H)_P$ will be denoted by $O(H)_1$, $SO(H)_1$ (resp. $O(H)_2$, $SO(H)_2$). It follows, from general principles, that the universal covering of $SO(H)_P$ is a Banach-Lie group and that the covering map is 2-sheeted. An explicit construction of the universal covering group $\text{Spin}(H)_1$ of $SO(H)_1$ has been given by P. de la Harpe [3]. Later, R.J. Plymen and R.F. Streater [9] gave an explicit construction of the universal covering group $\text{Spin}(H)_2$ of $SO(H)_2$. Both groups $\text{Spin}(H)_1$ and $\text{Spin}(H)_2$ will be called spinor groups and will be denoted by $\text{Spin}(H)$.

We denote by $SO(H)$ both the groups $SO(H)_1$ and $SO(H)_2$ and by $\rho : \text{Spin}(H) \to SO(H)$ the corresponding covering maps. In the following, we define the spin structures using the groups $\text{Spin}(H)$ and we give some properties of these structures. Some results about the Riemannian $P-$structures are given, too.

2 Definitions of the spin structures

All bundles, manifolds and morphisms considered in the following will be assumed of class C^∞. Let M be a connected and paracompact manifold,

modeled on a Banach space and let $\xi : E \to M$ be a vector bundle over M with fibre H.

Definition 2.1. A Riemannian $P-$structure on the vector bundle ξ is a reduction of its structural group to the group $O(H)_P$.

Remark 2.1. The existence of the Riemannian $P-$structures is a direct consequence of the fact that $GL(H)$ is contractible (Kuiper's theorem).

Definition 2.2. A Riemannian $P-$structure on the vector bundle ξ is said to be orientable if ξ admits a further reduction of its structural group to the group $SO(H)_{\hat{P}}$.

Theorem 2.1. *A Riemannian $P-$structure on the vector bundle ξ is orientable if and only if the first Stiefel-Whitney class $w_1(\xi)$, vanishes.*

Proof. The proof of proposition 6.2 from [6] can be repeated using the homomorphism $O(H)_P \to O(H)_P/SO(H)_P$. We give an alternative proof. The exact sequence

$$1 \to SO(H)_P \to O(H)_P \xrightarrow{p} Z_2 \to 1 \tag{2.1}$$

induces an exact sequence of the cohomology groups and sets ([5], 3.1 and 2.10.1)

$$0 \to H^1(M, SO(H)_P) \to H^1(M, O(H)_P) \xrightarrow{p^*} H^1(M, Z_2). \tag{2.2}$$

We denote by L the principal bundle of linear frames of ξ (for definition see Bourbaki [l]), interpreted as an element of $H^1(M, O(H)_P)$. From exactness of the sequence (2.2) it follows that the *Riemannian $P-$structure of ξ is orientable iff* $p^*(L) = 0$. Now we prove that $p^*(L) = w_1(\xi)$. By the naturally property of the characteristic classes, it is sufficient to do so when M is the classifying space BO of the group $O(H)_P$. But $H^1(BO, Z_2) = Z_2$ [6], hence the map p^* is either identically zero, or is the class w_1. The first alternative is not possible, because there exists at least a vector bundle with a non-orientable Riemannian $P-$structure (see Exemple 4 from [2]). For the theory of the characteristic classes considered here see U. Koschorke [6].

Corollary 2.1. *A connected and paracompact manifold N, modeled on H, endowed with a Riemannian $P-$structure is orientable with respect to this structure iff $w_1(N) = 0$.*

Theorem 2.2. *A connected and paracompact manifold N, modeled on H, is orientable with respect to all Riemannian $P-$structures which are compatible with its manifold structure if $H^1(N, Z_2) = 0$.*

Proof. It follows from a result of U. Koschorke [6, Proposition 6.3].

A reduction of the structural group of the vector bundle ξ to the group $O(H)_1$ (resp. $O(H)_2$) will be called a *Riemannian nuclear structure* (resp. a *Riemannian Hilbert-Schmidt structure*).

Let G be a Banach-Lie group and let $P(M, \pi, G)$ (where $\pi : P \to M$), be a principal bundle over M with group G. Let G' be another Banach-Lie group.

Definition 2.3. A principal bundle $P'(M, \pi', G')$ where $\pi' : P' \to M$ is said to be an extension of the principal bundle $P(M, \pi, G)$, associated to

the homomorphism $\varphi : G' \to G$ if there exists a morphism $\widetilde{\varphi} : P' \to P$ such that $(\widetilde{\varphi}, \varphi)$ is a morphism of principal bundles. We suppose that the vector bundle ξ is endowed with a reduction of its structural group to $SO(H)$ and we denote by $P(M, \pi, SO(H))$ the principal bundle of linear frames of it.

Definition 2.4. A spin structure on the vector bundle ξ, endowed with a reduction of its structural group to $SO(H)$, is an extension of principal bundle $P(M, \pi, SO(H))$, associated to the covering map ρ :Spin$(H) \to SO(H)$.

Such an extension will be denoted by $\Sigma(M, \pi,$Spin$(H))$ and will be called a *spin structure on* $P(M, \pi, SO(H))$ or a *spin structure on* M *with respect to* $P(M, \pi, SO(H))$, too.

The morphism $\widetilde{\rho} : \Sigma \to P$ is a 2-sheeted covering map and its restriction to fibres are 2-sheeted covering maps. For every a in Spin(H) and u in Σ, we have $\widetilde{\rho}(ua) = \widetilde{\rho}(u)\rho(a)$ and $\pi(\widetilde{\rho}(u)) = \pi'(u)$. It follows that the Definition 2.4 is equivalent to the following definition, given by A. Lichnerowicz [7] in a different context.

Definition 2.5. A spin structure on the vector bundle ξ, endowed with a reduction of its structural group to $SO(H)$, is a principal bundle $\Sigma(M, \pi, \text{Spin}(H))$ such that Σ is a 2-fold covering of P, the restriction of the covering map $\widetilde{\rho} : \Sigma \to P$ to fibres are 2-sheeted covering maps and $\widetilde{\rho}(ua) = \widetilde{\rho}(u)\rho(a)$, $\pi(\widetilde{\rho}(u)) = \pi'(u)$ hold, for every $a \in$Spin(H) and $u \in \Sigma$.

Remark 2.2. By a general result (see Bourbaki [1]), the principal bundle $P(M, \pi, SO(H))$ is determined (up to an isomorphism) by an open covering $\{U_i\}$ of M and a cocycle $g_{ij} : U_i \cap U_j \to SO(H)$. From Definition 2.4 it follows that, the principal bundle $\Sigma(M, \pi,$Spin$(H))$, when it exists, is determined by a cocycle $\widetilde{g}_{ij} : U_i \cap U_j \to$Spin$(H)$ such that $\rho(\widetilde{g}_{ij}) = g_{ij}$.

The following theorem gives another definition of the spin structures.

Theorem 2.3. *Let $P(M, \pi, SO(H))$ be the principal bundle of linear frames of ξ. The vector bundle ξ admits a spin structure if and only if there exists a cohomology class $\sigma \in H^1(P, Z_2)$ whose restriction to each fibre is non-zero.*

Proof. From the isomorphism $H^1(P, Z_2) \simeq$Hom$(H_1(P), Z_2)$ it follows that, there exists a not trivial homomorphism $\sigma : H_1(P) \to Z_2$. If $\varphi_1 : \pi(P) \to H_1(P)$ denotes Hurewicz's homomorphism, $\sigma \circ \varphi_1 : \pi_1(P) \to Z_2$ is an epimorphism, hence ker$(\sigma \circ \varphi_1)$ is a subgroup of index 2 in $\pi_1(P)$. Consequently, since the manifold P is locally contractible, there exists a covering space Σ of P such that $\widetilde{\rho}_*(\pi_1(\Sigma)) =ker(\sigma \circ \varphi_1)$, where $\widetilde{\rho} : \Sigma \to P$ is the covering map. The covering space Σ can be taken as the total space of a principal bundle over M with group Spin(H) which is an extension of $P(M, \pi, SO(H))$, therefore a spin structure of ξ.

Conversely, if $P(M, \pi, SO(H))$ admits a spin structure, $\Sigma(M, \widetilde{\pi},$Spin$(H))$, the total space Σ is a two-fold covering of P. Let s_0 and s_1 be two points in $\widetilde{\rho}^{-1}(u)$, where u is a fixed point in P. Denote by c a loop about u and by $\hat{c}$ its lift to Σ with $\hat{c}(0) = s_0$. The endpoint $\hat{c}(1)$ depends on $[c] \in \pi_1(P, u)$ the homotopy class of c. Define the homomorphism $\tau : \pi_1(P, u) \to Z_2$ by $\tau([c]) = 0$ if $\hat{c}(1) = s_0$ and $\tau([c]) = 1$ if $\hat{c}(1) = s_1$. Since Z_2 is commutative, τ vanishes on the commutator subgroup $[\pi_1(P), \pi_1(P)]$ of $\pi_1(P, u)$, therefore τ induces a homomorphism $\sigma : \pi_1(P, u)]/[\pi_1(P, u), \pi_1(P, u)] \to Z_2$. We can identify σ

with an element of $H^1(P, Z_2)$ via the isomorphisms $\pi_1(P,u)/[\pi_1(P),\pi_1(P)] \simeq H_1(P)$, $\mathrm{Hom}(H_1(P), Z_2) \simeq H^1(P, Z_2)$.

The exact sequence of groups

$$1 \to Z_2 \to \mathrm{Spin}(H) \to SO(H) \to 1 \tag{2.3}$$

induces an exact sequence of the cohomology groups and sets

$$0 \to H^1(M, Z_2) \to H^1(M, \mathrm{Spin}(H)) \to H^1(M, SO(H)) \xrightarrow{v} H^2(M, Z_2). \tag{2.4}$$

Denote by P the element of $H^1(M, SO(H))$ determined by $P(M, \pi, SO(H))$. Using the Remark 2.2. and the exactness of the sequence (2.4), we obtain

Theorem 2.4. *The vector bundle ξ admits a spin structure if and only if $v(P) = 0$.*

When $SO(H) = SO(H)_1$, P. de la Harpe [3] has proved that $v(P) = w_2(\xi)$, where $w_2(\xi)$ is the second Stiefel-Whitney class of ξ.

The following exact sequence

$$0 \to H^1(M, Z_2) \xrightarrow{i^*} H^1(SO(H), Z_2) \to H^2(M, Z_2), \tag{2.5}$$

where i is the natural inclusion of the fibre in the total space, can be obtained from spectral sequence associated to the principal bundle $P(M, \pi, SO(H))$ (see J.-P. Serre [11] p. 456).

If ξ admits a spin structure $\sigma \in H^1(P, Z_2)$, then $\sigma + \pi^*(b)$ where $b \in H^1(M, Z_2)$ is *the most general spin structure of* ξ. It follows that, there is a bijection between the set of isomorphism classes of spin structures of ξ and $H^1(M, Z_2)$. Consequently, a spin structure of ξ, is unique (up to an isomorphism) iff $H^1(M, Z_2) = 0$.

Theorem 2.5. *Let $\xi_1 \oplus \xi_2$ be the Whitney sum of the vector bundles ξ_1 and ξ_2 over M. Given spin structures on two of the three vector bundles $\xi_1, \xi_2, \xi_1 \oplus \xi_2$, there is a uniquely determined spin structure on the third.*

Proof. As in J. Milnor [8].

Let M' be another manifold and let $f : M' \to M$ be a morphism of manifolds. Denote by $Pf(M', \pi', SO(H))$ the principal bundle induced from $P(M, \pi, SO(H))$ by the map f. This principal bundle is determined (up to an isomorphism) by the cocycle $g_{ij} \circ f$ associated to the open covering $\{f^{-1}(U_i)\}$ where g_{ij} is the cocycle of $P(M, \pi, SO(H))$ associated to an open covering $\{U_i\}$. Using the Remark 2.2 we obtain

Theorem 2.6. *Let be $f : M' \to M$. If M admits a spin structure $\Sigma(M, \widetilde{\pi}, \mathrm{Spin}(H))$ with respect to $P(M, \pi, SO(H))$, then $\Sigma f(M', \pi', \mathrm{Spin}(H))$ is a spin structure on M' with respect to $Pf(M', \pi', SO(H))$.*

Suppose now that $f : M' \to M$ is a principal bundle with group G. It follows that, there is an action of G on Pf defined by $(p', u)g = (p'g, u)$ for $(p', u) \in Pf$ and $g \in G$.

Theorem 2.7. *Let be $f : M' \to M$ a principal bundle with group G. Then the following conditions are equivalent:*

(1) *There exists a spin structure $\Sigma(M, \widetilde{\pi}, \mathrm{Spin}(H))$ on $P(M, \pi, SO(H))$.*

(2) *There exists a spin structure $\Sigma'(M', \widetilde{\pi}', \mathrm{Spin}(H))$ on $Pf(M', \pi', SO(H))$ with the following properties:*

a) G acts on Σ' such that Σ'/G is a manifold and the projection $\Sigma' \to \Sigma'/G$ is a submersion,

b) The action of G on Σ' commutes with the action of $\mathrm{Spin}(H)$ *on* Σ'.

c) $\widetilde{\rho}'(wg) = \widetilde{\rho}'(w)g$ holds, for every $g \in G$ and $w \in \Sigma'$, where $\widetilde{\rho}' : \Sigma' \to Pf$ is the covering map.

Proof. (1) $\Rightarrow$ (2). By the Theorem 2.6, $\Sigma f(M', \pi', \mathrm{Spin}(H))$ is a spin structure on $Pf(M', \Pi', SO(H))$. Define an action of G on $\Sigma' = \Sigma f$ by $(p', v)g = (p'g, v)$, where $(p', v) \in \Sigma'$ and $g \in G$. This action is proper and free. Moreover, the map $g \to (p'g, v)$ is an immersion of G in Σ', since $f : M' \to M$ is a principal bundle. It follows that $\Sigma' \to \Sigma'/G$ is just a principal bundle (see Bourbaki [1]), hence the property a) is verified. From $(p', v)b = (p', vb)$ for $b \in$Spin(H), it follows $(p'g, vb) = (p', v)gb = (p', v)bg$ i.e. the property b).

For $w = (p', v) \in \Sigma$, we have $\widetilde{\rho}'(wg) = (p'g, \widetilde{\rho}(v)) = \widetilde{\rho}'(w)g$, i. e. the property c).

(2) $\Rightarrow$ (1) Define an action of Spin(H) on Σ'/G by $wGb = wbG$, where $w \in \Sigma'$, $b \in$Spin(H) and wG is the orbit of w, and a surjection $h : \Sigma'/G \to M$ by $h(wG) = f(\pi'(w))$. The local isomorphism $t : f^{-1}(U) \times \mathrm{Spin}(H) \to \Sigma'$, where U is an open subset of M, defines a local isomorphism $s : U \times \mathrm{Spin}(H) \to \Sigma'/G$ by

$$s(p, b) = t(p'G, b) = t(p', b)G, \text{ where } f(p') = p. \tag{2.6}$$

The last equality from (2.6) is a consequence of

$$\begin{cases} \pi'(\widetilde{\rho}'(wg)) = \pi'(\widetilde{\rho}'(w))g \\ \\ f^*(\widetilde{\rho}'(wg)) = f^*\widetilde{\rho}'(w), \quad g \in G, \ w \in \Sigma', \end{cases} \tag{2.7}$$

where $f^* : Pf \to P$ is the morphism induced by f. But (2.7) is equivalent to property c). It is not difficult to see that $h : \Sigma'/G \to M$ is a principal bundle with group Spin(H). The morphism $\widetilde{\rho} : \Sigma'/G \to P$ defined by $\widetilde{\rho}(wG) = f^*(\widetilde{\rho}'(w))$ satisfies $\pi \circ \widetilde{\rho} = h$ and $\widetilde{\rho}(wGb) = \widetilde{\rho}(wG)\rho(b)$, therefore $\Sigma'/G(M, h,$Spin$(H))$ is a spin structure on $P(M, \pi, SO(H))$.

Suppose that $f : M' \to M$ is a principal bundle with group Z_2. Let η denotes the involution of Pf defined by the action of Z_2 on it.

Corollary 2.7. *Let $f : M' \to M$ be a principal bundle with group Z_2. The following conditions are equivalent:*

(1) *There exists a spin structure $\Sigma(M, \pi, \mathrm{Spin}(H))$ on $P(M, \pi, SO(H))$,*

(2) *There exists a spin structure $\Sigma'(M', \widetilde{\pi}, \mathrm{Spin}(H))$ on $Pf(M', \pi', SO(H))$ endowed with an involution η' which corresponds to identity on* Spin(H) *and which commutes with the involution η.*

Proof. In order to apply the Theorem 2.7 it is sufficient to remark that η' defines an action of Z_2 on Σ', which commutes with the action of Spin(H), such that $\Sigma' \to \Sigma'/Z_2$ is a submersion and $\rho'(wZ_2) = \widetilde{\rho}'(w)Z_2$ holds, for every $w \in \Sigma'$.

Remark 2.3. The above corollary has been obtained by I. Popovici [10] in non-orientable and finite dimensional case.

Definition 2.6. Let N be a manifold modeled on H, with an oriented riemannian nuclear structure (resp. an oriented riemannian Hilbert-Schmidt structure). A spin structure on N is a spin structure on the tangent bundle TN.

3 Examples

a) The trivial bundle $M \times SO(H)$ admits a spin structure, unique if M is simply connected.

b) Let $\left\{ x = (x_1, x_2, ...)/x_i \in R,\ \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$ be the Hilbert space l_2. The Hilbert torus $T = l_2 / \sum_{i=1}^{\infty} Ze_i$, where $e_1, e_2, ...$ is a base of l_2, is a Hilbert-Lie group modelled on l_2. It admits a canonical analytic atlas such that the derivatives of the coordinate changes is always the identity on l_2. It follows that the corresponding Riemannian nuclear structure is orientable and T with this structure, admits a spin structure (with respect to $\text{Spin}(l_2)_1$) since $w_2(T) = 0$.

c) Let $M \subset H$, be a smoothly imbedded manifold with an oriented Riemannian nuclear structure. From Theorem 2.6, it follows that a spin structure on M determines a spin structure on the normal bundle to M and conversely.

4 Clifford bundles

In this section we limit our considerations to the group $SO(H)_1$. Let $Cl(H)$ be the Clifford algebra of H viewed as a C^*-algebra [3]. A $^*-$automorphism ψ of $Cl(H)$ which satisfies $\psi(H) = H$ is called a Bogoliubov automorphism. Denote by $\text{Bog}(Cl(H))$ the group of Bogoliubov automorphisms and by $C : O(H) \to \text{Bog}(Cl(H))$ the canonical isomporhism described in [3]. A Bogoliubov automorphism ψ is said to be *inner* if there exists $u \in Cl(H)$ such that $\psi(v) = uvu^{-1}$ for every $v \in Cl(H)$ and is said to be *inner and even* if u is even. Let $S(M, O(H))$ be a principal bundle. Its associated fibre bundle with fibre $Cl(H)$ (the action of the group $O(H)$ on $Cl(H)$ is given by C) is an algebric bundle, called the Clifford bundle. We can obtain the Clifford bundle in another way. For this, let be $\xi : E \to M$ a Riemannian vector bundle. The fibres of ξ are Hilbert spaces. Let E_p be the fibre of ξ in $p \in M$ and let $Cl(E_p)$ be the Clifford algebra of E_p. The set $\bigcup_{p \in M} Cl(E_p)$ and the projection $Cl(E_p) \to p$ can be taken as the total space and projection of the Clifford bundle. We remark that ξ can be identified with a subbundle of the Clifford bundle. Let $\text{Bog}(Cl(H))_i$ be the group of inner Bogoliubov automorphisms. The isomorphism $O(H)_1 \simeq \text{Bog}(Cl(H))_i$ (see [4]) implies the following

Theorem 4.1. *There exists a Riemannian nuclear structure on the Riemannian vector bundle $\xi : E \to M$ iff the Clifford bundle admits a reduction to* $\text{Bog}(Cl(H))_i$.

From the isomophism $SO(H)_1 \simeq \text{Bog}(Cl(H))_{ie}$ (see [3]), where $\text{Bog}(Cl(H))_{ie}$ is the group of inner and even Bogoliubov automorphism, it follows

Theorem 4.2. *A Riemannian nuclear structure on the Riemannian vector bundle $\xi : E \to M$ is orientable iff the Clifford bundle admits a reduction to* $\text{Bog}(Cl(H))_{ie}$.

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CONNEXIONS SUR LES FIBRES SPINORIELS

PAR

MIHAI ANASTASIEI

Dans [1] on a défini et étudié les structures spinorielles sur les variétés hilbertiennes. Dans le présent article on considère les connexions adaptées aux structures spinorielles (les connexions spinorielles). Après quelques résultats relatifs aux connexions sur les fibrés principaux banachique (§1), on définit les connexions spinorielles (§2). En utilisant la dérivée covariante des spineurs, on met en évidence une famille d'opérateurs différentiels du premier ordre (au sens de [7] p. 91) sur les fibrés spinoriels. Si la variété est a dimension finie l'operatéur de Dirac peut être déduit de cette famille des opérateurs.

1 Connexions sur les fibres principaux banachiques

Soient M une variété différentiable de classe C^∞ modelée sur un espace de Banach $\mathbf{M}$, G un groupe de Lie-Banach réel et $P(M, \pi_P, G)$ un fibré principal (abrégé f.p.) de base M, de groupe structural G et de projection π_P. Nous notons par $R_g : u \to ug$, $u \in P$, $g \in G$ l'action de G sur P et par $\sigma_u : G \to P$, $u \in P$, l'application $g \to ug$, $g \in G$. σ_u est un diffeomorphisme de classe C^∞ entre le groupe G et la fibre au-dessus du point $p = \pi_P(u)$. Le foncteur tangent sera noté par T. Considérons la suite exacte de fibrés vectoriels au-dessus de P:

$$0 \to P \times \mathbf{G} \xrightarrow{I} TP \xrightarrow{T\pi_P!} \pi_P^* TM \to 0 \tag{1.1}$$

où: - $\mathbf{G}$ est l'algèbre de Lie-Banach de G,
- π_P^*TM est le fibré image réciproque de TM par π_P,
- $T\pi_P! = (\tau_P, T\pi_P)$, $\tau_P : TP \to P$,
- $I(u, A) = T\tau_u(A)$, $u \in P$, $A \in \mathbf{G}$.

Un $G-$fibré vectoriel est un fibré vectoriel sur lequel G opère par des automorphismes de fibré vectoriel. Si nous considérons les actions naturelles de G sur TP et π_P^*TM ainsi que l'action de G sur $P\times\mathbf{G}$, $(u, A)g = (ug, ad(g-1)A)$, $u \in P$, $g \in G$, $A \in \mathbf{G}$, alors la suite (1.1) devient une suite exacte de $G-$fibrés vectoriels.

Définition 1.1. [8] Une connexion (infinitésimale) sur le f.p. $P(M, \pi_P, G)$ est une scission de la suite exacte (1.1) de $G-$fibrés vectoriels.

Soit $\Gamma : \pi_P^* TM \to TP$ une telle scission. Alors, il existe un morphisme unique $\omega : TP \to P \times \mathbf{G}$, tel que $\omega \circ I = \mathrm{id}|_{P\times\mathbf{G}}$. Pour chaque point $u \in P$, il existe une décomposition unique en somme directe de T_uP

$$T_uP = I(P \times G)_u \oplus \Gamma(\pi_P^* TM)_u. \tag{1.2}$$

La décomposition (1.2) definit de manière evidente deux projecteurs sur T_uP, qui seront notés par h (de noyau $I(P\times\mathbf{G})_u$) et v (de noyau $\Gamma(\pi_P^*TM)_u$). La compatibilité de Γ avec les actions de G sur π_P^*TM et TP implique

$$\Gamma(ug, Z) = TR_g\Gamma(u, Z),\ u \in P, g \in G, Z \in TM,\ \text{et} \tag{1.3}$$

$$\omega(TR_gX_u) = (u, \omega_u(X_u))g,\ u \in P,\ g \in G,\ X_u \in T_uP, \tag{1.4}$$

où nous avons noté par ω_u la restriction de ω à T_uP et nous avons posé $\omega(X_u) = (u, \omega_u(X_u))$. L'application linéaire $\omega_u : T_uP \to \mathbf{G}$, $u \in P$, a les deux propriétés suivantes

$$\omega_u(\sigma_u(A)) = A,\ A \in \mathbf{G}, \tag{1.5}$$

$$(R_g^*\omega)(X_u) \stackrel{def}{=} \omega_{ug}(TR_gX_u) = ad(g^{-1})\omega_u(X_u),\ u \in P, g \in G. \tag{1.6}$$

Soit $F_G(P)$ l'ensemble des fonctions différentiables de classe C^∞ définies sur P à valeurs dans $\mathbf{G}$ et soit $\omega : \mathcal{X}(P) \to F_G(P)$ une 1-forme de classe C^∞ sur P à valeurs dans $\mathbf{G}$, définie par:

$$\omega(X)_u = \omega_u(X_u),\ u \in P, X \in \mathcal{X}(P), \tag{1.7}$$

où $\mathcal{X}(P)$ est le module des champs de vecteurs sur P. Evidemment, (1.5) et (1.6) impliquent:

$$\omega(\sigma(A)) = A,\ A \in \mathbf{G}, \tag{1.8}$$

$$R_g^*\omega = ad(g^{-1})\omega,\ g \in G, \tag{1.9}$$

où $\sigma(A)$ est le champ vectoriel $u \to \sigma(A)$.

Réciproquement, une 1-forme sur P à valeurs dans $\mathbf{G}$, définit une application inverse à gauche pour I, compatible avec les actions de G sur $P \times \mathbf{G}$ et TP, grâce aux formules (1.8) et (1.9). Vu que la suite (1.1) est exacte, cette inverse définit une connexion sur $P(M, \pi_P, G)$. Donc, nous avons établi l'equivalence entre la définition 1.1 et la

Définition 1.2. Une connexion (infinitésimale) sur le f.p. $P(M, \pi_P, G)$ est une 1-forme sur P à valeurs dans G avec les propriétés (1.8) et (1.9).

Ainsi, nous avons recupéré, pour nos buts, une définition bien connue en dimension finie, des connexions (infinitésimales) sur un f.p. Une autre caractérisation est donnée par le

Theoreme 1.1. *L'existence sur un f.p. $P(M, \pi_P, G)$ d'une connexion (infinitésimale) équivaut à l'existence d'un projecteur $h : TP \to TP$ ($h \circ h = h$) avec les propriétés:*

$$\ker\ h = I(P \times \mathbf{G}), \tag{1.10}$$

$$TR_g \circ h = h \circ TR_g,\ \ g \in G. \tag{1.11}$$

La preuve de ce théorème est immédiate si nous remarquons que (1.3) équivaut à (1.11).

Le morphisme $F = v - h$ de TP définit une structure presque-produit sur P, associée d'une manière naturelle à la scission Γ. L'espace des vecteurs propres correspondant à la valeur propre 1 de l'opérateur $F_u : T_uP \to T_uP$ est $I(P \times \mathbf{G})_u$. Vu que le projecteur v a la propriété

$$TR_g \circ v = v \circ TR_g,\ \ g \in G, \tag{1.12}$$

il résulte que (1.11) équivaut à

$$TR_g \circ F = F \circ TR_g,\ \ g \in G. \tag{1.13}$$

En utilisant le théorème 1.1 on obtient le

Théorème 1.2. *Il existe une connexion (infinitésimale) sur $P(M, \pi_P, G)$ si et seulement si, il existe une structure presque-produit F sur P, avec les propriétés*

a) $F_u(X_u) = X_u \Leftrightarrow X_u \in I(P \times G)_u,\ X_u \in T_uP$,

b) $TR_g \circ F = F \circ TR_g,\ g \in G$.

Remarque. Les théorèmes 1.1 et 1.2 ont été établis en dimension finie, par V. Cruceanu dans [2] et [3].

Soit $(f, \varphi_0, h_0) : P(M, \pi_P, G) \to P'(M', \pi'_{P'}, G')$ où $f : P \to P'$, $\varphi_0 : G \to G'$, $h_0 : M \to M'$ un homomorphisme de f.p., c'est-à-dire:

$$\pi_{P'} \circ f = h_0 \circ \pi_P,\ \ f(ug) = f(u)\varphi_0(g),\ \ u \in P, g \in G. \tag{1.14}$$

Definition 1.3. Soient les f.p. $P(M, \pi_P, G)$ et $P'(M', \pi'_{P'}, G')$ munis des connexions (infinitésimales) Γ resp. Γ'. Nous dirons que l'homomorphisme (f, φ_0, h_0) est compatible avec les connexions Γ et Γ' si avons

$$Tf \circ \Gamma = \Gamma' \circ (f \times Th_0). \tag{1.15}$$

Remarque. La relation (1.15) équivaut à

$$\omega' \circ Tf = (f \times T\varphi_0) \circ \omega. \tag{1.16}$$

La démonstration du théorème suivant se conduit comme en dimension finie (voir [6] p. 79–82).

Théorème 1.3. *Soit* $(f, \varphi_0, h_0) : P(M, \pi_P, G) \to P'(M', \pi_{P'}, G')$ *un homomorphisme de fibrés principaux, avec* h_0 *un difféomorphisme.*

a) Soit Γ *une connexion (infinitésimale) sur* $P(M, \pi'_P, G)$*. Alors, il existe une connexion (infinitésimale) unique* Γ' *sur* $P'(M', \pi'_P, G')$ *de manière que l'homomorphisme* (f, φ_0, h_0) *soit compatible avec les connexions (infinitésimales)* Γ *et* Γ'.

b) En supposant que φ_0 *est un difféomorphisme local, soit* Γ' *une connexion (infinitésimale) sur* $P'(M', \pi_{P'}, G')$*. Alors, il existe une connexion (infinitésimale) unique* Γ *sur* $P(M, \pi_P, G)$ *telle que l'homomorphisme* (f, φ_0, h_0) *soit compatible avec les connexions (infinitésimales)* Γ *et* Γ'.

Soit $\mathbf{F}$ un espace de Banach. En supposant que G opère sur $\mathbf{F}$ par un homomorphisme $\psi : G \to GL(\mathbf{F})$, soit $\pi : E \to M$ le fibré vectoriel associé à $P(M, \pi_P, G)$ de fibre type $\mathbf{F}$. Par une modification légère d'une preuve de J-P. Penot (voir [8]) on peut montrer que toute connexion (infinitésimale) sur $P(M, \pi_P, G)$ induit une connexion vectorielle unique sur $\pi : E \to M$ c'est-à-dire il existe un morphisme de fibrés vectoriels $K : TE \to E$, tel que pour chaque carte vectorielle (U, φ, Φ) de π, nous avons

$$(\Phi \circ K \circ T\Phi^{-1}) = (x, \xi, y, \eta) = (x, \eta + \Gamma_\varphi(x))(y, \xi),\ x, y \in \mathbf{M} \tag{1.17}$$

$\xi, \eta \in \mathbf{F}$, où $\Gamma_\varphi(x) \in L^2(M, \mathbf{F}; \mathbf{F})$ correspond aux symboles de Christoffel usuels. Notons par $\mathcal{X}_E(M)$ le module des sections sur M dans le fibré vectoriel $\pi : E \to M$ et possons $\mathcal{X}_{TM}(M) = \mathcal{X}(M)$. Il existe (voir [4], p. 17) une dérivation covariante unique ∇_X, $X \in \mathcal{X}(M)$, associée naturellement à l'application K, donnée dans une carte vectorielle quelconque (U, φ, Φ) par

$$\nabla_X S|_{\varphi(p)} = \partial S_\varphi|_{\varphi(p)}(X_\varphi) + \Gamma_\varphi^{(x)}(X_\varphi, S_\varphi),\ p \in U, X \in \mathcal{X}(M), \tag{1.18}$$

$$x = \varphi(p),\ S \in \mathcal{X}_E(M),$$

où $X_\varphi = T_\varphi \circ X$, $S_\varphi = \Phi \circ S$ et ∂ est le symbole de différentiation de Fréchet. Supposons que M admet une partition de l'unité. D'après le lemme 3.1 de [4] et la formule (1.18) nous pouvons définir l'application $\nabla : T_pM \times \mathcal{X}_E(U) \to \mathcal{X}_E(U)$, avec U un ouvert de M par $(X_p, S) \to \nabla_{X_p} S = \nabla_X S$, où X est un champ arbitraire de vecteurs qui coïncide au point p avec X_p et $S \in \mathcal{X}_E(U)$. Cette application est $R-$linéaire relativement à X_p et

$$\nabla_{X_p}(fS) = T_pf(X_p)S + f(p)\nabla_{X_p}S,\quad S \in \mathcal{X}_E(M), \tag{1.19}$$

où f est une fonction réelle quelconque sur M. Nous considérons l'opérateur de différentiation covariante $\nabla : \mathcal{X}_E(M) \to \mathcal{X}_{L(TM,E)}(M)(S \to \nabla S)$ donne par

$$(\nabla S)(p) = (\nabla_{X_p} S)(p),\ p \in M, X_p \in T_pM,\ S \in \mathcal{X}_E(M). \tag{1.20}$$

En utilisant (1.19) on obtient le (voir aussi [4], p.6)

Théorème 1.1. *L'opérateur de différentiation covariante* ∇ *est un opérateur différentiel du premier ordre.*

2 Structures spinorielles et connexions spinorielles

Soit $\mathbf{H}$ un espace de Hilbert réel, séparable et de dimension infinie. Nous notons par $Cl(\mathbf{H})_\infty$ l'algèbre de Clifford sur $\mathbf{H}$ rélative à la forme quadratique $Q(x) = \|x\|^2 = (x, x)$, $x \in \mathbf{H}$, structuré comme une C^*-algebre (voir [5]). Soit $\mathbf{J}$ une structure complexe sur $\mathbf{H}$, c'est-à-dire un opérateur orthogonal sur $\mathbf{H}$ avec $\mathbf{J}^2 = -\mathrm{id}$. Si nous posons i $x = \mathbf{J}x$ et $\langle x, y\rangle = (x, y) + (\mathbf{J}x, y)$, $x, y \in \mathbf{H}$, l'espace H devient un espace de Hilbert complexe, qui sera noté par $\mathbf{H}_c$. Soient $\Lambda^n \mathbf{H}_C$ la puissance extérieure des n exemplaires de $\mathbf{H}_C$ et $\Lambda(\mathbf{H}_C) = \subset_{n\geq 0} \oplus \Lambda^n \mathbf{H}_C$ avec la structure naturelle de l'espace de Hilbert complexe.

L'algèbre extérieure $\Lambda(\mathbf{H}_C)$ a une Z_2-graduation naturelle, $\Lambda(\mathbf{H}_C) = \Lambda^0 \oplus \Lambda^1$ où $\Lambda^0 = \oplus_{k\geq 0} \Lambda^{2k} \mathbf{H}_C$ et $\Lambda^1 = \oplus_{k\geq 0} \Lambda^{2k+1} \mathbf{H}_C$. Il existe une représentation fidèle et irréductibile F de $Cl(\mathbf{H})_\infty$ sur l'espace de Hilbert $\Lambda(\mathbf{H}_C)$ (voir par exemple [9]). Soient $Cl(\mathbf{H})^*_\infty$ le groupe multiplicatif des éléments inversibles de $Cl(\mathbf{H})_\infty$ et $\mathrm{Spin}(\mathbf{H})_\infty = \{u \in Cl(\mathbf{H})^*_\infty \mid u\mathbf{H}u^{-1} = \mathbf{H}, u\beta(u) = \beta(u)u = 1, \alpha(u) = u\}$ où α est l'involution et β est l'antiinvolution principale de $Cl(\mathbf{H})_\infty$. P. la Harpe a montré dans [5] que le groupe $\mathrm{Spin}(\mathbf{H})_\infty$ est groupe de Lie-Banach.

Soit $O(\mathbf{H})_1$ le groupe des opérateurs orthogonaux sur $\mathbf{H}$ de la forme id.$+A$, où A est un opérateur nucléaire. Le groupe de Lie-Banach $O(\mathbf{H})_1$ a deux composantes connexes. Le revêtement universel de la composante connexe de l'identité $SO(\mathbf{H})_1$, est exactement $\mathrm{Spin}(\mathbf{H})_\infty$ (voir [5]). Soit $\Delta = F \mid \mathrm{Spin}(\mathbf{H})_\infty$. Les espaces Λ^0 et Λ^1 sont invariants par Δ (voir [9]) et ils définissent deux sous-représentations de Δ qui seront notées par Δ^0 et Δ^1, respectivement. Ces deux représentations Δ^0 et Δ^1 sont continues, injectives et irréductibles (voir [9]). L'espace $\Lambda(\mathbf{H}_C)$ (en abrégé Λ) s'appelle l'espace de spineurs relatif à $SO(\mathbf{H})_1$ et les espaces Δ^0 et Δ^1 s'appellent les espaces de semi-spineurs relatifs à $SO(\mathbf{H})_1$.

Soit $P(M, SO(\mathbf{H})_1)$ un f.p. de base M et de groupe structural $SO(\mathbf{H})_1$. Un f.p. $\Sigma(M,\mathrm{Spin}(\mathbf{H})_\infty)$ qui est l'extension de $P(M, SO(\mathbf{H})_1)$ associée à l'homomorphisme de revêtement $\rho : \mathrm{Spin}(\mathbf{H})_\infty \to SO(\mathbf{H})_1$, s'appelle structure spinorielle sur $P(M, SO(\mathbf{H})_1)$ ou structure spinorielle sur M relative à $P(M, SO(\mathbf{H})_1)$ (voir [1]). Nous notons par $(\widetilde{\rho}, \rho) : \Sigma(M,\mathrm{Spin}(\mathbf{H})_\infty) \to P(M, SO(\mathbf{H})_1)$ l'homomorphisme d'extension.

Definition 2.1. On appele *connexion spinorielle* une connexion (infinitésimale) sur $\Sigma(M,\mathrm{Spin}(\mathbf{H})_\infty)$.

Soient $C(P)$ et $C(\Sigma)$ les ensembles de connexions sur $P(M, SO(\mathbf{H})_1)$ et $\Sigma(M, \mathrm{Spin}(\mathbf{H})_\infty)$, respectivement. Comme ρ est un difféomorphisme local il résulte en vertu du théorème 1.3, qu'il existe une bijection $\widetilde{\rho} : C(\Sigma) \to C(P)$. Si nous notons par $\widetilde{\omega}$ et ω les 1-formes des deux connexions correspondantes par $\widetilde{\rho}$, nous avons:

$$\omega_{\widetilde{\rho}(\widetilde{u})}(T\widetilde{\rho}X_{\widetilde{u}}) = T\rho\widetilde{\omega}_{\widetilde{u}}(X_{\widetilde{u}}),\ \widetilde{u} \in \Sigma,\ X_{\widetilde{u}} \in T_{\widetilde{u}}\Sigma. \tag{2.1}$$

$\widetilde{\omega}_{\widetilde{u}}(X_{\widetilde{u}})$ et $\omega_{\widetilde{\rho}(\widetilde{u})}(T\widetilde{\rho}X_{\widetilde{u}})$ sont dans les algèbres de Lie-Banach des groupes $\mathrm{Spin}(H)_\infty$ et $SO(H)_1$, respectivement.

En vertu de la proposition 12 [5] nous obtenons

$$\|\omega_{\widetilde{\rho}(\widetilde{u})}(T\widetilde{\rho}X_{\widetilde{u}})\|_1 = 4\|\widetilde{o}o_{\widetilde{u}}(X_{\widetilde{u}})\|_\infty, \quad \widetilde{u}\in\Sigma, X_{\widetilde{u}}\in T_{\widetilde{u}}\Sigma \tag{2.2}$$

où $\| \ \|_1$ est la norme nucléaire et $\| \ \|_\infty$ est la norme de C^*-algèbre sur $Cl(\mathbf{H})_\infty$. En dimension finie la relation (2.2) coincide avec la relation (5.4) de [10].

Vu que Spin$(\mathbf{H})_\infty$ opère sur Λ nous pouvons considérer le fibré associé à $P(M,$Spin$(\mathbf{H})_\infty)$ avec la fibré type Λ qui sera nommé fibré spinoriel. Les fibrés associés à $P(M,$ Spin$(\mathbf{H})_\infty)$ avec les fibrés type Λ^0 et Λ^1, respectivement, seront nommés fibrés semi-spinoriels. Une section du fibré spinoriel sera nommée champ spinoriel.

Une connexion spinorielle induit une dérivation covariante dans le fibré spinoriel qui sera nomée dérivation spinorielle. L'existence de la bijection $\widetilde{\rho}$ implique que toute connexion sur $P(M,SO(\mathbf{H})_1)$ induit une dérivation spinorielle.

Une réduction de groupe structural de TM (M modélée sur $\mathbf{H}$) au groupe $SO(\mathbf{H})_1$ s'appelle structure riemanniene nucléaire orientée sur M. Si M est munie d'une structure riemannienne nucléaire orientée, le fibré de repères de TM est un f.p. de base M et de groupe structural $SO(\mathbf{H})_1$ qui sera noté par $R(M)$. Nous supposons que M a une structure spinorielle, c'est-à-dire il existe un f.p. $\Sigma(M)$ de base M et de groupe structural Spin$(\mathbf{H})_\infty$, l'extension de $R(M)$ par ρ.

Dans la suite nous allons mettre en évidence une classe d'opérateurs différentiels du premier ordre sur une variété munie d'une structure spinorielle. Soit $\Lambda(M)$ le fibré spinoriel. Vu que $H\subset Cl(\mathbf{H})_\infty$, pour tout $x\in\mathbf{H}$, on a une application linéaire $F(x):\Lambda\to\Lambda$. Pour $b\in$Spin$(\mathbf{H})_\infty$ et $s\in\Lambda$ nous avons

$$\begin{aligned}\Delta(b)(F(x)s) = F(b)(F(x)s) = F(bx)s = F(bxb^{-1}b)s =\\ = F(bxb^{-1})F(b)s = F(\rho(b)x)F(b)s.\end{aligned} \tag{2.3}$$

Soit (U_i,φ_i) un atlas de la variété M et $\{\tau_i : \tau_s^{-1}(U_i)\to U_i\times\Lambda\}$ les cartes du fibré spinoriel $\tau_s:\Lambda(M)\to M$. Les applications $\tau_j\circ\tau_i^{-1}: U_i\cap U_j\to L(\Lambda)$ ont leur images dans $\Delta(\text{Spin}(H)_\infty)$. Vu que Δ est injective, $\tau_j\circ\tau_i^{-1}(p)$, $p\in U_i\cap U_j$, s'identifie à son image dans Spin$(H)_\infty$. Soient $\{\Phi_i:\tau^{-1}(U_i)\to U_i\times H\}$ les cartes du fibré tangent $\tau:TM\to M$. Les applications $\Phi_j\circ\Phi_i^{-1}$: $U_i\cap U_j\to L(H)$ ont leur images dans $SO(H)_1$; comme ρ est surjectif il existe $b\in$Spin$(H)_\infty$ tel que $\rho(b)=\Phi_j\circ\Phi_i^{-1}(p)=\partial(\varphi_j\circ\varphi_i^{-1})(p)$, $p\in U_i\cap U_j$. Nous définissons maintenant une application $\Psi: TM\times\Lambda(M)\to\Lambda(M)$ par

$$\Psi(X_p,s_p)(p) = \tau_{i,p}^{-1}(F(X_{\varphi_i}))(s_{\tau_i}),\ p\in M, X_p\in T_pM, s_i\in\Lambda_pM \tag{2.4}$$

où $X_{\varphi_i}=\Phi_{i,p}(X_p)$, $s_{\tau_i}=\tau_{i,p}(s_p)$.

D'après la relation (2.1) il résulte que la définition de Ψ ne dépend pas des cartes locales choisies. Une connexion linéaire sur M induit une connexion sur $\Lambda(M)$ et donc une dérivation covariante ∇. Pour tout $X\in\mathcal{X}(M)$ nous définissons un opérateur $D_X:\mathcal{X}_{\Lambda(M)}(M)\to\mathcal{X}_{\Lambda(M)}(M)$ par

$$(D_XS)(p) = \Psi(X_p,\nabla_{X_p}S),\ p\in M,\ S\in\mathcal{X}_{\Lambda(M)}(M) \tag{2.5}$$

Théorèm 2.1. *a) L'opérateur D_X est pour tout $X \in \mathcal{X}(M)$ un opérateur différentiel du premier ordre.*

b) Le symbole de l'opérateur D_X est donné par

$$\sigma_1(D_X)(v_p) = v_p(X_p)\Psi(X_{p,\cdot}) \tag{2.6}$$

où v_p est une 1-forme non nulle sur T_pM pour chaque $p \in M$.

Démonstration. a) Soit $S \in \mathcal{X}_{\Lambda(M)}(M)$ tel que le jet d'ordre 1, $(j^1S)(p) = 0$. Vu que ∇ est un opérateur différentiel du premier ordre, il résulte que $\nabla_{X_p}S = 0$, donc $(D_XS)(p) = 0$.

b) Soient f une fonction réelle sur M telle que $f(p) = 0$ et $T_pf = v_p$. Soit $S \in \mathcal{X}_{\Lambda(M)}(M)$ tel que $S(p) = s$. Nous avons

$$\sigma_1(D_X)(v_p)(s) = D_X(fS)(p) = \Psi(X_p, \nabla_{X_p}(fS)) = \Psi(X_p, T_pf(X_p)S(p)+$$
$$+f(p)\nabla_{X_p}S) = \Psi(X_p, v_p(X_p)s) \text{ donc } \sigma_1(D_X)(v_p) = v_p(X_p)\Psi(X_{p,\cdot}).$$

Un calcul direct montre que

$$\sigma_1(D_X)(v_p) \circ \sigma_1(D_X)(v_p) = \alpha[v_p(X_p)]^2\mathrm{id}$$

où α est un nombre réel non nul et X_p est non nul. Il résulte que le symbole $\sigma_1(D_X)(vp)$ est injectif seulement si v_p est injective, donc l'opérateur D_X n'est pas elliptique au sens de [7].

Supposons que M est de dimension finie égale à n et introduisons l'opérateur de Dirac D à l'aide des opérateurs D_X (voir aussi [7]). Soit U un voisinage ouvert de $p \in M$ muni d'un champ de repères orthonormés $\{X_1, X_2, ..., X_n\}$. Définissons d'abord un opérateur différentiel du premier ordre D_U sur $\Lambda(M)$ par

$$(D_US)(p) = \sum_{k=1}^{n}(D_{X_k}S)(p), \quad S \in \mathcal{X}_{\Lambda(M)}(M). \tag{2.7}$$

La définition de D_U ne dépend pas du champ $\{X_1, ..., X_n\}$. Si l'on considère une famille d'opérateurs D_{U_i} de la forme (2.5) avec $\{U_i\}$ un recouvrement ouvert de M, deux opérateurs arbitraires D_{U_i} et D_{U_j} coïncident sur $U_i \cap U_j$. Il résulte que la famille d'opérateurs D_{U_i} définit un opérateur différentiel du premier ordre unique D sur M tel que la restriction de D à chaque U_i coïncide avec D_{U_i}. Le symbole de l'opérateur D est donné par

$$\sigma_1(D)(v_p) = \sum_{k=1}^{n} v_p(X_{k,p})\Psi(X_{k,p,\cdot}). \tag{2.8}$$

Parce que le champ $\{X_1, ..., X_n\}$ est un champ de repères orthonormés nous obtenons

$$\sigma_1(D)(v_p) \circ \sigma_1(D)(v_p) = \sum_{k=1}^{n}[v_p(X_{k,p})]^2\mathrm{id}. \tag{2.9}$$

Par suite l'opérateur D est elliptique.

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Faculté de Mathématique
Université de Jassy, R.S. România

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CONSTANT LINEAR CONNECTIONS ON BANACH MANIFOLDS

BY

M. ANASTASIEI and IULIAN POPOVICI

The notion of constant linear connection was studied by G. Vranceanu and others from various points of view. An approach of the second author to this subject is used to define and study this notion in the category of analytic Banach manifolds.

Introduction

An affine connection on an open subset U of $\mathbb{R}^n$ is well-determined by n^3 real functions Γ^k_{ij} $(i, j, k = 1, ..., n)$, defined on U. G. Vranceanu has considered the affine connection defined by $\Gamma^k_{ij} =$ constant on U and called it constant affine connection. By a remark of G. Vranceanu and Gr.C. Moisil, in this case Γ^k_{ij} define on an $n-$dimensional vector space a structure of an $n-$dimensional algebra $\mathbf{A}$ and conversely, the constants of structure of such an algebra $\mathbf{A}$ define a constant affine connection ∇ on an open subset of $\mathbf{R}^n$ [7]. In this way there appears a correspondence $\nabla \to \mathbf{A}$ studied in detail by G. Vranceanu [7], [8] and others. We quote the following interesting result: the constant affine connection ∇ is plate if and only if the algebra $\mathbf{A}$ is commutative and associative.

The second author of this paper succeeded to give a global form of this notion and to obtain the global forms of the old results and some new results [5], [6]. His approach to this subject can be used to extend the notion of constant connection to Banach manifolds. This is the purpose of the present paper.

Firstly, some facts about Banach manifolds and linear connections on such manifolds are given.

The notion of constant linear connection is defined for Banach manifolds of class C^∞. A theorem which shows that the natural place of this concept is the category of analytic Banach manifolds is proved.

Finally, a splitting of the Banach manifolds with constant linear connections is given and a generalization of the result quoted above is proved.

1 PRELIMINARIES AND NOTATIONS

Let M be a paracompact Banach manifold of class C^∞ modeled by the Banach space $\mathbf{M}$. Assume that the norm of $\mathbf{M}$ is of class C^∞ on $\mathbf{M}-\{0\}$. It follows that $\mathbf{M}$ admits a C^∞partition of unity. We remark that the assumption concerning the norm of $\mathbf{M}$ is fulfilled if it originates in an inner product on $\mathbf{M}$, therefore if $\mathbf{M}$ is a Hilbert space. Let us denote by $\mathcal{F}(M)$ the ring of real functions of class C^∞ on M and by $\mathcal{X}(M)$ the $\mathcal{F}(M)-$module of sections of class C^∞ of the tangent bundle TM.

Let $X \in \mathcal{X}(M)$ be a vector field on M and let (U,φ) be a local chart around of $p \in M$. The local section $X|_U$ is well-defined by a $C^\infty-$map X_φ: $\varphi(U) \to \mathbf{M}$, called the local representation of X. We put $X_{\varphi(p)} = X_\varphi(\varphi(p))$. For another local chart (V,ψ) around of p, the local representation X_ψ of X is given by

$$X_{\psi(p)} = D_{\varphi(p)}(\psi\circ\varphi^{-1})(X_{\varphi(p)}), \tag{1.1}$$

where $D_{\varphi(p)}(\psi\circ\varphi^{-1})$ is the Fréchet derivative of $\psi\circ\varphi^{-1}$ in the point $\varphi(p) \in \varphi(U\cap V)$. Let Y be another vector field on M. The bracket $[X,Y]$ is a vector field whose local representation is given by (see [4])

$$[X,Y]_{\varphi(p)} = D_{\varphi(p)}X_\varphi(Y_{\varphi(p)}) - D_{\varphi(p)}Y_\varphi(X_{\varphi(p)}). \tag{1.2}$$

Given a local chart (U,φ), we denote by $K(U,\varphi)$ the set of those vector fields on U, whose local representations are constant. The set $K(U,\varphi)$ has the following two properties.

(1.3) The map $K(U,\varphi) \to T_pM$ given by $X \to X_p$ is an isomorphism of vector spaces for every $p \in U$.

(1.4) The bracket $[X,Y]=0$ for every $X,Y \in K(U,\varphi)$.

Every $X \in \mathcal{X}(M)$ generates a local 1-parameter group α_t, of diffeomorphisms of M. A vector field Y is said to be invariant by X if $\alpha_{t,*}Y = Y$. The Lie derivative of Y with respect to X is given by $(L_XY)_p = \lim\limits_{t\to 0}(Y_p(\alpha_{t,*}Y_p))\cdot t^{-1}$, therefore Y is invariant by X if and only if $L_XY = [X,Y]=0$. We can say that $K(U,\varphi)$ is a set of vector fields on U which are invariant by each other.

Let $\{(U_i,\varphi_i)\}$ be the complete atlas of M. By a linear connection Γ on M we shall understand (see also [2]) a local connector on M, i.e. a collection of $C^\infty-$maps $\Gamma_{\varphi_i} : \varphi_i(U_i) \to L^2(\mathbf{M};\mathbf{M})$ such that

$$\begin{aligned}\Gamma_{\varphi_j(p)} = D_{\varphi_i(p)}(\varphi_j\circ\varphi_i^{-1})\circ[D^2_{\varphi_j(p)}(\varphi_i\circ\varphi_j^{-1})+ \\ +\Gamma_{\varphi_i(p)}(D\varphi_{j(p)}(\varphi_\circ\varphi_j^{-1}), D_{\varphi_j(p)}(\varphi_i\circ\varphi_j^{-1})]\end{aligned} \tag{1.3}$$

holds for $p \in U_i\cap U_j \neq \emptyset$, where $\Gamma_{\varphi_i(p)} = \Gamma_{\varphi_i}(\varphi_i(p))$.

Given $X,T \in \mathcal{X}(M)$, condition (1.5) assures that

$$\nabla_XY \overset{def}{=} D_{\varphi_i(p)}Y_{\varphi_i}(X_{\varphi_i(p)}) + \Gamma_{\varphi_i(p)}(X_{\varphi_i(p)}, Y_{\varphi_i(p)}) \tag{1.4}$$

defines a new vector field on M which is denoted by ∇_XY and is called the covariant derivative of Y in the direction of X. The map $\nabla : \mathcal{X}(M)\times$

$\mathcal{X}(M) \to \mathcal{X}(M)$ given by $(X,Y) \to \nabla_X Y$ is linear in the first variable and satisfies

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \quad X,Y,Z \in \mathcal{X}(M), \tag{1.5}$$

and

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y, \quad f \in \mathcal{F}(M), \tag{1.6}$$

therefore it is a covariant differentiation on M.

The torsion and the curvature of Γ are given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.7}$$

and

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \ X,Y,Z \in \mathcal{X}(M) \tag{1.8}$$

respectively.

S. Kobayashi and K. Nomizu have defined generalized affine connections as connections in principal fibre bundle of affine frames over M. They proved that there exists a one-to-one correspondence between the set of generalized affine connections and the set of pairs (Γ, K), where Γ is a linear connection and K is a tensor field of type $(1,1)$ (see Ch. III, §3 of [3]). The generalized affine connection which corresponds to (Γ, I), where I is the tensor of Kronecker, was called the affine connection associated to Γ. This logical distinction between a linear connection and an affine connection can also be made in our context (see [1]). Moreover, the theorem which says that an affine connection is plate (cf. Ch. II, §9 of [3]) if and only if $R = 0$ and $T = 0$ is still true.

Let C be a vector field on M. A linear connection Γ is said to be invariant by C if

$$[C, \nabla_X Y] = \nabla_X[C,Y] + \nabla_{[C,X]}Y, \text{ holds for every } X,Y \in \mathcal{X}(M). \tag{1.9}$$

It follows from (1.11) that a linear connection Γ is invariant by C if and only if $\nabla_X Y$ is invariant by C when X and Y are invariant by C.

Finally, we remark that, with minor changes, the results what follow are true without hypothesis of paracompactness of manifolds and even in the case "no Hausdorf".

2 CONSTANT LINEAR CONNECTIONS

Definition 2.1. Let M be a Banach manifold of class C^∞. A linear connection Γ on M is said to be *constant with respect to the local chart* (U,φ) of M, if Γ is invariant by every vector field from $K(U,\varphi)$ i.e.

$$\nabla_X Y \in K(U,\varphi) \text{ for all } X,Y \in K(U,\varphi). \tag{2.1}$$

Suppose that Γ is constant with respect to (U,φ). Then a new operation can be defined on $K(U,\varphi)$ by $XY = \nabla_X Y$, $X,Y \in K(U,\varphi)$ or

$$X_{\varphi(p)}Y_{\varphi(p)} = \Gamma_{\varphi(p)}(X_{\varphi(p)}, Y_{\varphi(p)}). \tag{2.2}$$

It follows from continuity of the bilinear map $\Gamma_{\varphi(p)}$ that

$$\|X_{\varphi(p)}Y_{\varphi(p)}\| \leq |\Gamma_{\varphi(p)}| \; \|X_{\varphi(p)}\| \; \|Y_{\varphi(p)}\|,$$

where $\|\cdot\|$ is the norm on $\mathbf{M}$ and the $|\cdot|$ is the norm on $L^2(\mathbf{M};\mathbf{M})$, therefore $K(U,\varphi)$ with the product defined by (2.2) is a Banach algebra isomorphic to $\mathbf{M}$ as normed linear spaces via the isomorphism $T_pM \cong \mathbf{M}$. We denote this Banach algebra by $A(\Gamma, K(U,\varphi))$.

Definition 2.2. Let $\mathbf{A}$ be a Banach algebra. A linear connection Γ is said to be *constant* on a manifold M if there exists an atlas $\alpha = \{(U_i,\varphi_i)\}$ on M such that Γ to be constant with respect to each (U_i,φ_i) and $A(\Gamma, K(U_i,\varphi_i))$ to be isomorphic to $\mathbf{A}$ as Banach algebras. The triplet $(M,\Gamma,\mathbf{A})$ will be called a *Vranceanu's space*. The atlas α will be called an *atlas adapted* to $(M,\Gamma,\mathbf{A})$ and the atlas α completed with all local charts with the properties required above will be denoted by α^* and will be called *complete atlas adapted* to $(M,\Gamma,\mathbf{A})$.

Proposition 2.1. *Let $(M,\Gamma,\mathbf{A})$ be a Vranceanu's space. The linear connection Γ is symmetric $(T = 0)$ if and only if $\mathbf{A}$ is commutative.*

Proof. The local representation of the torsion T in a local chart (U,φ) is $T(X,Y)_{\varphi(p)} = \Gamma_{\varphi(p)}(X_{\varphi(p)}, Y_{\varphi(p)}) - \Gamma_{\varphi(p)}(Y_{\varphi(p)}, X_{\varphi(p)})$, therefore Γ is symmetric if and only if $\Gamma_{\varphi(p)}$ are symmetrical maps. It follows that Γ is symmetric if and only if $A(\Gamma, K(U,\varphi))$ is commutative, therefore if and only if $\mathbf{A}$ is commutative. Q.E.D.

In Definition 2.2 the manifold M was assumed of class C^∞. The following theorem shows that a manifold of class C^∞ with a constant linear connection has a structure of analytic manifold.

Theorem 2.1. *Let M be a Banach manifold of class C^∞ and let Γ be a linear connection on M which is constant with respect to an atlas $\alpha = \{(U_i,\varphi_i)\}$ and with a Banach algebra $\mathbf{A}$. Then M has a structure of analytic manifold $'M$ given by α and Γ induces on $'M$ an analytic connection $'\Gamma$.*

Proof. If Γ is constant on M with respect to α and $\mathbf{A}$, the maps Γ_{φ_i} are necessarily constant maps. Relation (1.5) can be written as follows

$$D^2_{\varphi_j(p)}(\varphi_i \circ \varphi_j^{-1}) = D_{\varphi_j(p)}(\varphi_i \circ \varphi_j^{-1}) \circ \Gamma_{\varphi_i(p)} - \Gamma_{\varphi_i(p)}(D_{\varphi_j(p)}(\varphi_i \circ \varphi_j^{-1}), D_{\varphi_j(p)}(\varphi_i \circ \varphi_j^{-1})). \tag{2.3}$$

Let us note $u = D(\varphi_i \circ \varphi_j^{-1}) : \varphi_j(U_i \cap U_j) \to L(\mathbf{M};\mathbf{M})$. Then (2.3) becomes

$$D_{\varphi_j(p)}u(h) = u_{\varphi_i(p)}(\Gamma_{\varphi_j(p)}(h,h)) - \Gamma_{\varphi_i(p)}(u_{\varphi_j(p)}(h), u_{\varphi_j(p)}(h)), \; h \in \mathbf{M} \tag{2.4}$$

where $u_{\varphi_j(p)} = u(\varphi_j(p))$. We put $x = \varphi_j(p)$. The map u is of class C^∞ by hypothesis of the theorem. Let us consider the Taylor series of u in a neighborhood of x (see [4], Ch. I, §4)

$$u(x) + D_x uh + D_x^2 uh^2 + ... + D_x^n uh^n + ..., \tag{2.5}$$

where $h^k = (h,h,...,h) \in \mathbf{M}^k$.

We shall prove that series (2.5) converges in a small neighborhood of x. Firstly, on differentiating by $n-$times the function u and using (2.4) we obtain

$$
\begin{aligned}
&D_x^n u(h_1, ..., h_n) = D_x^{n-1} u(h_1, ..., h_{n-2}, \Gamma_{\varphi_j(p)}(h_{n-1}, h_n)) - \\
(2.6) \qquad & - \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} \Gamma_{\varphi_i(p)}(D_x^k u(h_1, ..., h_k), D_x^{n-k-1} u(h_{k+1}, ..., h_n)),
\end{aligned}
$$

where $(h_1, ..., h_n) \in \mathbf{M}^n$. In what follows all norms will be denoted by $|\cdot|$ and the index x will be omitted since all derivatives are in the point x. Equation (2.4) leads to

$$
|Duh| \leq |u|\ |\Gamma_{\varphi_j(p)}|\ |h|^2 + |\Gamma_{\varphi_i(p)}|\ |u|^2 |h|^2,\ h \in \mathbf{M}. \tag{2.7}
$$

By a well-known definition $|Duh| = \sup\limits_h \{|Duh|,\ |h| \leq 1\}$. Using (2.7) we arrive at

$$
|Du| \leq |u|\ |\Gamma_{\varphi_j(p)}| + |\Gamma_{\varphi_i(p)}|\ |u|^2, \tag{2.8}
$$

$$
|Du| \leq \lambda |u| \text{ where } \lambda = |\Gamma_{\varphi_j(p)}| + |u|\ |\Gamma_{\varphi_i(p)}|. \tag{2.9}
$$

Now we prove by mathematical induction

$$
\frac{1}{n!} |D^n u| \leq \lambda^n |u|. \tag{2.10}
$$

From (2.6) it follows

$$
\begin{aligned}
&|D^n u(h_1, ..., h_n)| \leq |D^{n-1} u|\ |\Gamma_{\varphi_j(p)}|\ |h_1| ....|h_n| + \\
(2.11) \qquad & + \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} |\Gamma_{\varphi_i(p)}|\ |D^k u|\ |D^{n-k-1} u|\ |h_1|...|h_n|.
\end{aligned}
$$

Using $|D^n u| = \sup\limits_{(h_1 ... h_n)} \{|D^n u(h_1, ...h_n),\ |h_i| \leq 1, i = 1, ..., n\}$ and, the inductive hypothesis we obtain

$$
|D^n u| \leq |D^{n-1} u|\ |\Gamma_{\varphi_j(p)}| + |\Gamma_{\varphi_i(p)}| \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} |D^k u|\ |D^{n-k-1} u| \leq
$$

$$
\leq (n-1)! \lambda^{n-1} |u|\ |\Gamma_{\varphi_j(p)}| + |\Gamma_{\varphi_i(p)}| \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} k!(n-k-1)! \lambda^{n-1} |u|^2,
$$

hence

$$
\frac{1}{n!} |D^n u| \leq |u| \lambda^{n-1} \left(\frac{1}{n} |\Gamma_{\varphi_j(p)}| + |\Gamma_{\varphi_i(p)}|\ |u| \right) \leq \lambda^n |u|.
$$

The series (2.5) converges if and only if the series $\sum_{n\geq 0} \frac{1}{n!}|D^n uh^n|$ converges. Using (2.10) we obtain

$$\frac{1}{n!}|D^n uh^n| \leq \frac{1}{n!}|D^n u|\ |h|^n \leq |u|(\lambda|h|)^n. \tag{2.12}$$

Consequently, by comparison test, series (2.5) converges for $|h| \leq 1/\lambda$. It follows that u is analytic i.e. $\{(U_i, \varphi_i)\}$ defines on M a structure of analytic manifold. Q.E.D.

3 SOME TYPES of VRANCEANU'S SPACES

Let $\mathbf{A}$ be a Banach algebra. The law of product on $\mathbf{A}$ defines an element $B \in L^2(\mathbf{A}; \mathbf{A})$ putting $xy = B(x, y)$, $x, y \in \mathbf{A}$. Conversely, every element of $L^2(\mathbf{A}; \mathbf{A})$ defines a law of product on the Banach space $\mathbf{A}$ which changes it into a Banach algebra. Let $(M, \Gamma, \mathbf{A})$, where M is an analytic manifold, be a Vranceanu's space and let $\{(U_i, \varphi_i)\}$ be an atlas adapted to it. The isomorphism of $A(\Gamma, K(U_i, \varphi_i))$ to $\mathbf{A}$ leads via the isomorphisms $A(\Gamma, K(U_i, \varphi_i)) \cong T_pM$ and $T_pM \cong \mathbf{M}$, to an isomorphism of normed linear spaces $\theta_i : \mathbf{M} \to \mathbf{A}$ such that $B(\theta_i u, \theta_i v) = \theta_i \Gamma_{\varphi_i}(u, v)$ for $u, v \in \mathbf{M}$. The isomorphism θ_i depends on (U_i, φ_i) but it does not depend on the points of U_i.

Now, let Γ be a certain linear connection on M and let $\{(V_j, \psi_j)\}$, be an analytic atlas of M. Suppose that for each chart (V_j, ψ_j) there exists an isomorphism of normed linear spaces $\theta_j : \mathbf{M} \to \mathbf{A}$ (θ_j does not depend on points of V_j) such that $B(\theta_j u, \theta_j v) = \theta_j \Gamma_{\psi_j(p)}(u, v)$, $u, v \in \mathbf{M}$. Then Γ_{ψ_j} is constant on V_j and $A(\Gamma, K(V_j, \psi_j))$ is isomorphic to $\mathbf{A}$, i.e. the triplet $(M, \Gamma, \mathbf{A})$ is a Vranceanu's space. Therefore we have proved

Proposition 3.1. *A triplet $(M, \Gamma, \mathbf{A})$ is a Vranceanu's space if and only if there exists an atlas $\{(V_j, \psi_j)\}$ on M such that for each (V_j, ψ_j) there exists an isomorphism $\theta_j : \mathbf{M} \to \mathbf{A}$ satisfying*

$$B(\theta_j u, \theta_j v) = \theta_j \Gamma_{\psi_j(p)}(u, v), \quad u, v \in \mathbf{M}. \tag{3.1}$$

Remarks. 1) The isomorphisms θ_j are determined up to an isomorphism of the Banach algebra $\mathbf{A}$, i.e. if θ_j satisfies (3.1), then $h \circ \theta_j$, where h is an isomorphism of $\mathbf{A}$, satisfies (3.1), too.

2) The atlas $\beta = \{(V_j, \psi_j)\}$ from the above proposition is an atlas adapted to $(M, \Gamma, \mathbf{A})$. It will be called $\theta-$atlas and completed with all charts which satisfy (3.1) will be denoted by β^* and will be called the complete $\theta-$atlas of $(M, \Gamma, \mathbf{A})$.

3) If (V_j, ψ_j) is a chart from β^*, then $(V_j, g \circ \psi_j)$, where $g \in GL(\mathbf{M})$, satisfies (3.1) because we can write $\theta_{gj} = \theta_{gj} \circ \theta_j^{-1} \circ \theta_j$, where $\theta_{gj} : \mathbf{M} \to \mathbf{A}$ corresponds to $(V_j, g \circ \psi_j)$ and $\theta_{gj} \circ \theta_j^{-1}$ is an isomorphism of $\mathbf{A}$. It follows

that $\Gamma_{g\circ\psi_j}$ is constant on V_j. This shows that if we add to β^* all charts of the form $(V_j, g\circ\psi_j)$ with g from $GL(\mathbf{M})$ and (V_j,ψ_j) from β^*, we obtain the complete atlas adapted to $(M,\Gamma,\mathbf{A})$ (denoted above by α^*).

Using Proposition 3.1 we obtain the following corollary of Theorem 2.1.

Corollary 3.1. *Let M be a Banach manifold of class C^∞ equipped with a linear connection Γ and let $\mathbf{A}$ be a Banach algebra. If there exists an atlas $\{(V_j,\psi_j)\}$ with the property that for each chart (V_j,ψ_j) there exists an isomorphism of normed linear spaces $\theta_j : \mathbf{M}\to\mathbf{A}$ such that (3.1) to be true, then $\{(V_j,\psi_j)\}$ gives to M a structure of analytic manifold.*

Definition 3.1. An atlas $\{(U_i,\psi_i)\}$ of the analytic manifold M is said to be *affine* if $\varphi_j\circ\varphi_i^{-1} : \varphi_i(U_i\cap U_j)\to\varphi_j(U_i\cap U_j)$ has the form

$$(\varphi_j\circ\varphi_i^{-1})(u) = S(u)+u_0, \quad \text{where } u_0, u\in\mathbf{M} \text{ and } S\in GL(\mathbf{M})), \tag{3.2}$$

for all pairs (i,j) with $U_i\cap U_j\neq\emptyset$.

We remark that in this case $D(\varphi_j\circ\varphi_i^{-1}) = S$ and $D^2(\varphi_j\circ\varphi_i^{-1}) = 0$. Conversely, an atlas $\{(U_i,\varphi_i)\}$ on M which satisfies

$$D^2(\varphi_j\circ\varphi_i^{-1}) = 0 \text{ on } U_i\cap U_j\neq\emptyset \text{ for all pairs } (i,j), \tag{3.3}$$

is affine because the general solution of equation (3.3) is (3.2).

Let $(M,\Gamma,\mathbf{A})$ be a Vranceanu's space and let $\{(V_i,\psi_i)\}$ be a $\theta-$atlas. Suppose that $\{(V_i,\psi_i)\}$ is affine, therefore $D(\psi_j\circ\psi_i^{-1}) = S_{ji}\in GL(\mathbf{M})$. Then (3.2) becomes

$$D_{\psi_i(p)}(\psi_j\circ\psi_i^{-1})(\Gamma_{\psi_i}(u,v)) = \Gamma_{\psi_j}(D_{\psi_i(p)}(\psi_j\circ\psi_i^{-1})u, D_{\psi_i(p)}(\psi_j\circ\psi_i^{-1})v), \tag{3.4}$$

or

$$S_{ji}(\Gamma_{\psi_i}(u,v)) = \Gamma_{\psi_j}(S_{ji}u, S_{ji}v), \quad u,v\in\mathbf{M}. \tag{3.5}$$

Using (3.1) we obtain

$$S_{ji}\theta_i^{-1}B(\theta_i u,\theta_i v) = \theta_j^{-1}B(\theta_j S_{ji}u, \theta_j S_{ji}v), \quad u,v\in\mathbf{M}. \tag{3.6}$$

If we put $u=\theta_i^{-1}u'$, $v=\theta_i^{-1}v'$, in (3.6) we arrive at

$$\theta_j S_{ji}\theta_i^{-1}B(u',v') = B(\theta_j S_{ji}\theta_i^{-1}u', \theta_j S_{ji}\theta_i^{-1}v') \ \ u',v'\in\mathbf{A}, \tag{3.7}$$

therefore $\overline{S}_{ji} = \theta_j S_{ji}\theta_i^{-1}$ is an isomorphism of $\mathbf{A}$.

We denote by G the subset of $GL(\mathbf{M})$ whose elements are of the form

$$S_{ji} = \theta_j^{-1}\overline{S}_{ji}\theta_i, \text{ where } \overline{S}_{ji} \text{ is an isomorphism of } \mathbf{A}. \tag{3.8}$$

Theorem 3.1. *Let M be an analytic Banach manifold endowed with an affine atlas $\beta=\{(U_i,\varphi_i)\}$ and let $\mathbf{A}$ be a Banach algebra. If assume that for each chart (U_i,φ_i) there exists an isomorphism of normed spaces $\theta_i : M\to A$*

which does not depend on points from U_i then the following statements are equivalent:

a) there exists a linear connection Γ such that $(M, \Gamma, \mathbf{A})$ is a Vranceanu's space with β as $\theta-$atlas.

b) every change of charts from β is the composition of a translation on $\mathbf{M}$ *and an element of G.*

Proof. Assuming a), since β is affine, for (U_i, φ_i) and (U_j, φ_j) with $U_i \cap U_j \neq \emptyset$ we have $(\varphi_j \circ \varphi_i^{-1})(u) = S_{ji}(u) + u_0$, therefore $\varphi_j \circ \varphi_i^{-1} = T_{u_0} \circ S_{ji}$, where $T_{u_0}(u) = u + u_0$ is the translation by u_0. By the considerations made above, $S_{ji} \in G$, therefore b) follows.

Let us suppose b). For each (U_i, φ_i), we define Γ_{φ_i} by

$$\Gamma_{\varphi_i}(u, v) = \theta_i^{-1} B(\theta_i u, \theta_i v) \quad u, v \in \mathbf{M}. \tag{3.9}$$

Let us prove that $\{\Gamma_{\varphi_i}\}$ is a local connector. Since β is affine we must verify (3.5), where $S_{ji} \in G$. If we replace Γ_{φ_i} and Γ_{φ_j} given by (3.9) in (3.5) we obtain (3.6) which is equivalent to (3.7). But (3.7) is true because $S_{ji} \in G$. From (3.9) and Proposition 3.1 it follows that $(M, \Gamma, \mathbf{A})$ is a Vranceanu's space with β as $\theta-$atlas. Q.E.D.

Remark. The connection Γ from a) of Theorem 3.1 is unique by Proposition 3.1.

Definition 3.2. A Vranceanu's space $(M, \Gamma, \mathbf{A})$ will be called of the *first kind*, *second kind* or *third kind* if the complete atlas α^* adapted to it, satisfies the following conditions, respectively:

1) α^* is affine,

2) α^* is not affine but contains an affine atlas of M,

3) a^* does not contain any affine atlas of M.

Using Proposition 3.1 and the remark which follows it we obtain

Proposition 3.2. *Let β be a $\theta-$atlas of the Vranceanu's space $(M, \Gamma, \mathbf{A})$. Then $(M, \Gamma, \mathbf{A})$ is of the first kind, second kind or third kind if β satisfies 1), 2) or 3) from Definition 3.2, respectively.*

On $\mathbf{A}$ we can consider a new structure of Banach algebra given by

$$B^s(x, y) = \frac{1}{2}(B(x, y) + B(y, x)), \quad x, y \in \mathbf{A}. \tag{3.10}$$

We denote this new Banach algebra by ${}^s\mathbf{A}$ and we remark that ${}^s\mathbf{A}$ is commutative.

Let Γ be a certain linear connection on M and let $\{\Gamma_{\varphi_i}\}$ be its local connector. For each φ_i let ${}^s\Gamma_{\varphi_i}$ be given by

$${}^s\Gamma_{\varphi_i}(u, v) = \frac{1}{2}[\Gamma_{\varphi_i}(u, v) + \Gamma_{\varphi_i}(v, u)] \quad u, v \in M. \tag{3.11}$$

It is easy to check that $\{{}^s\Gamma_{\varphi_i}\}$ is a local connector. We denote by ${}^s\Gamma$ the linear connection given by $\{{}^s\Gamma_{\varphi_i}\}$ and by ${}^s\nabla$ the covariant differentiation associated to ${}^s\Gamma$. It follows easily

$$2{}^s\nabla_X Y = \nabla_X Y + \nabla_Y X - [X, Y] \quad X, Y \in \mathcal{X}(M). \tag{3.12}$$

Proposition 3.3. *If the triplet* $(M, \Gamma, \mathbf{A})$ *is a Vranceanu's space then* $(M,^s\Gamma,^s\mathbf{A})$ *is also a Vranceanu's space. Moreover, we have*

1) If $(M,^s\Gamma,^s\mathbf{A})$ *is of the first kind (third kind) then* $(M, \Gamma, \mathbf{A})$ *is also of the first kind (third kind).*

2) If $(M, \Gamma, \mathbf{A})$ *is of the second kind, then* $(M,^s\Gamma,^s\mathbf{A})$ *is of the second kind.*

Proof. Let $\beta = \{(U_i, \varphi_i)\}$ a $\theta-$atlas of $(M, \Gamma, \mathbf{A})$, therefore $\theta_i \Gamma_{\varphi_i}(u, v) = B(\theta_i u, \theta_i v)$ for every i and $u, v \in \mathbf{M}$. It follows easily that $\theta_i\ {}^s\Gamma_{\varphi_i}(u, v) = B^s(\theta_i u, \theta_i v)$, therefore $(M,^s\Gamma,^s\mathbf{A})$ is a Vranceanu's space with β as $\theta-$atlas. Let β^* and ${}^s\beta^*$ be the complete $\theta-$atlas of $(M, \Gamma, \mathbf{A})$ and $(M,^s\Gamma,^s\mathbf{A})$, respectively. From $\beta^* \subset^s \beta^*$ and Proposition 3.1 follow easily 1) and 2). Q.E.D.

Remark. The inclusion $\beta^* \subset^s \beta^*$ shows also that if $(M, \Gamma, \mathbf{A})$ is of the first kind, then $(M,^s\Gamma,^s\mathbf{A})$ is of the first kind or of the second kind. Also, if $(M, \Gamma, \mathbf{A})$ is of the third kind, then $(M,^s\Gamma,^s\mathbf{A})$ if of the second kind or of the third kind.

The local representation of the curvature tensor of a linear connection Γ on M is given by

$$R_x(u, v)w = D_x\Gamma_{\varphi_i}(u)(v, w) - D_x\Gamma_{\varphi_i}(v)(u, w) +$$

$$+\Gamma_x(u, \Gamma_x(v, w)) - \Gamma_x(v, \Gamma_x(u, w)) \tag{3.13}$$

$$u, v, w \in \mathbf{M},\ x = \varphi_i(p),\ p \in M.$$

Proposition 3.4. *Let* $(M, \Gamma, \mathbf{A})$ *be a Vranceanu's space. The affine connection* Γ' *associated to* Γ *is plate if and only if* $\mathbf{A}$ *is associative and commutative.*

Proof. Assume that Γ' is plate. This is equivalent to $T = 0$ and $R = 0$. From $T = 0$ it follows $\Gamma_x(u, v) = \Gamma_x(v, u)$ for $u, v \in \mathbf{M}$. From $R = 0$ it follows

$$\Gamma_x(u, \Gamma_x(v, w)) = \Gamma_x(v, \Gamma_x(u, w))\ u, v, w \in \mathbf{M}. \tag{3.14}$$

Using (3.1) we obtain $B(u', v') = B(v', u')$ and $B(u', B(v', w')) = B(v', B(u', w'))$, where $u' = \theta_i u$, $v' = \theta_i v$, $w' = \theta_i w$. Using the first, which says that $\mathbf{A}$ is commutative, in the second we obtain $B(u', B(w', v')) = B(B(u', w'), v')$ i.e. $\mathbf{A}$ is associative. Conversely, if $\mathbf{A}$ is commutative and associative, using (3.1) we obtain easily that Γ_{φ_i} are symmetrical and (3.14) i.e. Γ' is plate. Q.E.D.

Remark. Proposition 3.4 is the generalization of a result due to G. Vranceanu [8].

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Faculty of Mathematics
"Al. I. Cuza" University of Iasi
and
Institute of Mathematics
Bucharest

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
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GENERALIZED AFFINE CONNECTIONS ON BANACH MANIFOLDS*

BY

M. ANASTASIEI

The theory of nonlinear connections in the category of Banach vector bundles has been developed by J. Vilms ([7], [8]). A class of nonlinear connections, called *homogeneous connections*, is of great importance in the theory of Finsler connections ([3], [5]).

The purpose of this paper is the study of another class of nonlinear connections, called *generalized affine connections* (g.a.c., for short). The term agrees with the one used in [4, p. 127]. In the first section some new results regarding the nonlinear connections are given. The second section contains the definition of g.a.c. and some of their properties (associated linear connections, geodesics and others). The flat g.a.c. are studied in the third section.

1 Nonlinear connections

Let M be a paracompact manifold of class C^∞ (smooth), modeled by the Banach space $\mathbf{M}$ and let $p : E \to M$ be a smooth vector bundle of fiber type a Banach space $\mathbf{E}$. Denote by $p^{-1}TM \to E$ the pull-back by p of the tangent bundle $\pi : TM \to M$ and by $p! = (Tp, \tau)$, where Tp is the tangent map to p and $\tau : TE \to E$ is the tangent bundle to the manifold E. The map $Tp : TE \to TM$ gives to TE a second (different) structure of vector bundle.

A smooth nonlinear connection is a smooth splitting of the following exact sequence

$$0 \to VE \xrightarrow{i} TE \xrightarrow{p!} p^{-1}TM \to 0 \tag{1.1}$$

of vector bundles over E. Here $VE := \ker(p!) = \ker(Tp)$ denotes the vertical subbundle of TE and i is the inclusion map.

The vertical subbundle $VE \to E$ is canonically isomorphic to $p^{-1}E \to E$ (the pull-back of E by p). Hence, there exists a canonical morphism (over P) $r : VE \to E$ of vector bundles, isomorphic on the fibres.

A splitting of the exact sequence (1.1), i.e. a nonlinear connection is given by a smooth morphism $V : TE \to VE$, such that $V \circ i = \mathrm{id}|VE$, or

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equivalently, a smooth morphism $W : p^{-1}TM \to TE$ such that $p! \circ W = \mathrm{id}|p^{-1}TM$. Moreover, we have $i \circ V + W \circ p! = \mathrm{id}|TE$. This implies $TE = VE + HE$, where $HE = \ker V = \mathrm{im} W$. Obviously, HE is isomorphic to $p^{-1}TM$ as vector bundles. The morphism (over p) $K := r \circ V : TE \to E$ is called the connection map and $v = i \circ V$, $h = W \circ p!$ are called vertical and horizontal projections, respectively. The morphism $J = i \circ p!$ of TE satisfies $J^2 = 0$ since $p! \circ i = 0$, therefore J defines an almost tangent structure on E. Obviously, $JV(E) = 0$ and $\mathrm{Im} J = VE$. The morphism $\gamma = 2hI$, where I is the identity on TE, satisfies

$$J \circ \gamma = J, \ \gamma \circ J = -J. \tag{1.2}$$

Conversely, a morphism γ satisfying (1.2) determines a unique splitting of the exact sequence (1.1), i.e a nonlinear connection on $p : E \to M$. Indeed, let W' be any right splitting map of the sequence (1.1) (W' exists if M admits smooth partitions of unity). We put $W - hW'$, where $2h = I + \gamma$. The morphism W does not depend on W' and it is easy to check, using (1.2), that $p! \circ W = \mathrm{id}_{p^{-1}TM}$. Therefore, we have the following definition of the nonlinear connections, equivalently to that previously given.

Definition 1.1. A nonlinear connection on $p : E \to M$ is a smooth morphism γ of TE (over $\mathrm{id}|_E$) satisfying (1.2).

The Definition 1.1 generalizes a definition of nonlinear connections on finite dimensional manifolds given by J. Grifone [3]. As in finite dimensional case (see [3]), one can proves the following.

Theorem 1.1. *A smooth morphism γ of TE is a nonlinear connection on $p : E \to M$ if and only if it defines an almost product structure on $E(\gamma \circ \gamma = I)$ such that for every $u \in E$, the eigenspace of γ_u (the restriction of γ to $p^{-1}u$)) which corresponds to the eigenvalue 1 be V_uE.*

2 Generalized affine connections

Let $\mathbf{F}$ be a Banach space. The map $\to: \mathbf{F} \times \mathbf{F} \to \mathbf{F}$ given by $(u, v) \to \overrightarrow{uv} = v - u$ defines the so-called canonical affine structure on $\mathbf{F}$. Every vector bundle can be considered as an affine bundle if one considers its fibers with the canonical affine structure.

Let $\mathbf{F}'$ be another Banach space. A map $t : \mathbf{F} \to \mathbf{F}'$ is said to be affine if $t(u) = T(u) + t(0)$ for every $u \in \mathbf{F}$, where $T : \mathbf{F} \to \mathbf{F}'$ is a linear map. If we regard $\mathbf{F}$ and $\mathbf{F}'$ as affine spaces, the map t is affine if and only if it is an affine morphism.

Given two vector bundles $E \to M$ and $E' \to M'$, a map $h : E \to E'$ which preserves the fibers is said to be affine if it is smooth and its restrictions to fibers are affine. Of course, h can be considered as a morphism in the category of affine bundles.

Definition 2.1. A nonlinear connection on $p : E \to M$ will be called generalized affine connection (briefly g.a.c.) if its connection map, denoted above by K, is an affine map with respect to the structure of vector bundle of TE given by $Tp : TE \to TM$.

An examination of the local situation will be suitable to lead us to the essential properties of g.a.c. Let (U, φ) be a local chart on M. We identify U with $\varphi(U)$ and, restricting U if necessary, suppose that there exists a

bundle chart $U \times \mathbf{E} \cong \mathbf{E}|_U$. Then the tangent map gives a local chart $U \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \cong TE|_U$ and the sequence (1.1) restricted to U becomes

$$0 \to U \times \mathbf{E} \times 0 \times \mathbf{E} \xrightarrow{i} U \times \mathbf{E} \times \mathbf{M} \times \mathbf{E} \xrightarrow{p!} U \times \mathbf{E} \times \mathbf{M} \to 0, \tag{2.1}$$

where $p!(x, a, \lambda, b) = (x, a, \lambda)$, $x \in U$, $\lambda \in \mathbf{M}$, $a, b \in \mathbf{E}$.

The map T_p is locally given by $T_p(x, a, \lambda, b) = (x, \lambda)$. Therefore the fibers of bundle $T_p : TE \to TM$ are isomorphic to $x \times \mathbf{E} \times \lambda \times \mathbf{E} \cong \mathbf{E}^2$. J. Vilms has proved (see [7]) the following

Lemma. *A morphism (over p) $K : TE \to E$ is the connection map of a nonlinear connection on $p : E \to M$, if and only if it is locally given by*

$$K(x, a, \lambda, b) = (x, b + \omega(x, a)\lambda), \ \ x \in U, \ \ \lambda \in \mathbf{M}, \ \ a, b \in \mathbf{E}, \tag{2.2}$$

where $\omega : U \times \mathbf{E} \to L(\mathbf{M}, \mathbf{E})$ is smooth.

For the above the nonlinear connection we shall prove

Lemma 2.1. *A morphism (over p) $K : TE \to E$ is the connection map of a g.a.c. if and only if it is locally given by*

$$K(x, a, \lambda, b) = (x, b + \Gamma(x)(a, \lambda) + A(x)\lambda), \tag{2.3}$$

where $\Gamma : U \to L^2(\mathbf{E}, \mathbf{M}; \mathbf{E})$ and $A : U \to L(\mathbf{M}, \mathbf{E})$ are smooth maps.

Proof. Let K be the connection map of a g.a.c. By Definition 2.1. the map $(x, a, \lambda, b) \to (x, b + \omega(x, a)\lambda)$ must be affine on $Tp-$fibers. Consequently, the map $(a, b) \to b + \omega(x, a)\lambda$ of $\mathbf{E} \times \lambda \times \mathbf{E} \to \mathbf{E}$ must be affine with respect to both the variables. Being linear, hence affine with respect to b, it remains to be affine with respect to a. This happens if and only if there exists a smooth map $\widetilde{\omega} : U \to L(\mathbf{E}, L(\mathbf{M}, \mathbf{E}))$ such that $\omega(x, a) = \widetilde{\omega}(x)(a) + \omega(x, 0)$. We put $A(x) = \omega(x, 0)$. Since $L(\mathbf{E}, L(\mathbf{M}, \mathbf{E})) \cong L^2(\mathbf{E}, \mathbf{M}; \mathbf{E})$, $\widetilde{\omega}$ determines a unique smooth map $\Gamma : U \to L^2(\mathbf{E}, \mathbf{M}; \mathbf{E})$ such that $\widetilde{\omega}(x)(a)\lambda = \Gamma(x)(a, \lambda)$. Therefore, $\omega(x, a)\lambda = \Gamma(x)(a, \lambda) + A(x)\lambda$ and (2.3) follows from (2.2).

The maps Γ and A will be called local components of the g.a.c..

Remarks. If the connection map of a nonlinear connection is linear on $Tp-$fibers, the connection becomes a linear connection. A g.a.c. is linear if and only if A vanishes on U.

When $\omega(x, a)$ is 1-homogeneous with respect to a, or equivalently, K is 1-homogeneous on $Tp-$fibers, the nonlinear connection is called homogeneous connection. The class of homogeneous connections is used in the theory of Finsler connections (see [3], [5]). In the definition of an homogeneous connections one needs a greater generality, namely the smoothness of it must be assumed only on $E - 0$, otherwise it becomes linear. Our considerations from the first section remain true in such a generality (with the appropriate modifications), but it is not necessary for the theory of g.a.c.

Let (U, φ) and (V, ψ) two local charts on M, such that $U \cap V \neq \emptyset$. We put $f = \psi \circ \varphi^{-1}$. If $\Phi : p^{-1}(U) \to U \times \mathbf{E}$ and $\Psi : p^{-1}(V) \to V \times \mathbf{E}$ are bundle local charts, we denote by $B : U \cap V \to L(\mathbf{E}, \mathbf{E})$ the map $x \to B(x) = \Psi \circ \Phi^{-1}$. In this notations the change of bundle local charts on E can be written as $(x, a) \to (f(x), B(x)a)$, $a \in E$ and the change of bundle local charts on $Tp : TE \to TM$ induced by it, is given by $(x, a, \lambda, b) \to (f(x)), B(x)a, \partial f(x), \partial B(x)(\lambda)a + B(x)b)$, $x, \lambda \in \mathbf{E}$, $a, b \in \mathbf{E}$, where ∂ means Fréchet differentiation. Let us denote by $\overline{\Gamma}$ and $\overline{A}$ the local

components of g.a.c. with respect to the local chart (V,ψ). Using (2.3) and the expressions of changes of bundle local charts given above, we find the following transformation rule for the local components of a g.a.c.

$$\begin{aligned}&\overline{\Gamma}(f(x))(B(x)a,\partial f(x)\lambda)+A(f(x))\partial f(x)=B(x)\Gamma(x)(a,\lambda)+\\ &+B(x)A(x)\lambda-\partial B(x)(\lambda)a,\quad x\in U\cap V,\ a,b\in\mathbf{E},\ \lambda\in\mathbf{M}.\end{aligned}\tag{2.4}$$

For $a=0$, the relation (2.4) becomes

$$A(f(x))\partial f(x)\lambda=B(x)A(x)\lambda,\quad x\in U\cap V,\quad \lambda\in\mathbf{M},\tag{2.5}$$

which, used in (2.4), leads to

$$\overline{\Gamma}(f(x))(B(x)a,\partial f(x)\lambda)=B(x)\Gamma(x)(a,\lambda)-\partial B(x)(\lambda)a.\tag{2.6}$$

The relation (2.5) shows that A is the local part of a section, denoted also by A, of the vector bundle $L,(TM,E)\to M$ (of fiber $L(T_xM,E_x)$, $x\in M$). The relation (2.6) is just the transformation rule of the local connector of a linear connection on E. Therefore, Γ defines a linear connection on E, which will be denoted also by Γ. Conversely, a section A of the vector bundle $L(TM,E)\to M$ and a linear connection Γ on E determine a unique g.a.c. via their local parts. So, we have proved

Theorem 2.1. *Let $p:E\to M$ be a Banach vector bundle. There exists a one-to-one correspondence between the set of g.a.c. on $p:E\to M$ and the pairs (Γ,A), where Γ is a linear connection on $p:E\to M$ and A is a section of $L(TM,E)\to M$.*

Let us denote by $\mathcal{X}_E(M)$ the set of smooth sections of $p:E\to M$ and let us put $\mathcal{X}(M)=\mathcal{X}_{TM}(M)$. Now, let us regard $p:E\to M$ as an affine bundle. Its fiber in $x\in M$ will be denoted by ${}^aE_x$ and x, identified with zero of E_x will be called the *contact point* of M with ${}^aE_x$. A map $P:M\to E$ given by $x\to P_x\in{}^a E_x$ is by definition of class C^∞, if the map $a:M\to E$ defined by $x\to a_x=\overrightarrow{xP_x}\in E_x$ is of class C^∞.

Such a map P of class C^∞ will be called a point field. We denote by $\mathcal{P}_E(M)$ the set of point fields of class C^∞ and we put $\mathcal{P}(M)=\mathcal{P}_{TM}(M)$.

By Theorem 2.1 a g.a.c. is well determined by the pair (Γ,A). But a linear connection, defines a covariant differentiation i.e. a map $\nabla:\mathcal{X}(M)\times\mathcal{X}_E(M)\to\mathcal{X}_E(M)$ with the following properties:

$$\nabla(X,\alpha a)=\alpha\nabla(X,a)+X(\alpha)\cdot a,\tag{2.7}$$

$$\nabla(X,a+b)=\nabla(X,a)+\nabla(X,b),\tag{2.8}$$

$$\nabla(\alpha X+\beta Y,a)=\alpha\nabla(X,a)+\beta\nabla(Y,a)\quad X,Y\in\mathcal{X}(M),\ a,b\in\mathcal{X}_E(M).\tag{2.9}$$

and $\alpha,\beta\in\mathcal{F}(M)$ the module of real functions defined on M.

Using ∇ and A we shall define an analogue of V for a g.a.c., namely: $D:\mathcal{P}(M)\times\mathcal{P}_E(M)\to\mathcal{P}_E(M)$ given by

$$\overrightarrow{QD(P,Q)}=\nabla(X,a)+A(X),\ \text{where } X=\overrightarrow{xP}\text{ and } a=\overrightarrow{xQ}.\tag{2.10}$$

Theorem 2.2. *The map D associated to a g.a.c. as above, has the following properties:*

$$D(P, \alpha Q + \beta R) = \alpha D(P, Q) + \beta D(P, R) + X(\alpha)P + X(\beta)Q \tag{2.11}$$

$$D(\alpha P + \beta P', Q) = \alpha D(P, Q) + \beta D(P', Q), \tag{2.12}$$

(2.13)
$$D(x, Q) = Q, \text{ for } \alpha, \beta \in \mathcal{F}(M),\ \alpha + \beta = 1,\ P \in \mathcal{P}(M),\ Q, R \in \mathcal{P}_E(M)$$

and $X = \overrightarrow{xP}$ where x is the contact point field.
Proof. To prove (2.11) we remark that it is equivalent to

$$(*)\quad \overrightarrow{xD(P, \alpha Q + \beta R)} = \alpha \overrightarrow{xD(P, Q)} + \beta \overrightarrow{xD(P, R)} + X(\alpha)\overrightarrow{xP} + X(\beta)\overrightarrow{xQ},$$

or

$$\overrightarrow{(\alpha Q + \beta R)D(P, \alpha Q + \beta R)} = \alpha \overrightarrow{QD(P, Q)} + \beta \overrightarrow{RD(P, R)} + X(\alpha)\overrightarrow{xP} + X(\beta)\overrightarrow{xQ}.$$

Now we can use (2.10) to obtain
(**)
$$\nabla(X, \alpha a + \beta b) + A(X) = \alpha \nabla(X, a) + \alpha A(X) + \beta \nabla(X, b) + \beta A(X) + X(\alpha)a + X(\beta)b,$$

where $a = \overrightarrow{xQ},\ b = \overrightarrow{xR}$.

Since $\alpha + \beta = 1$, (**) is true by virtue of (2.7) and (2.8). The proof of (2.11) follows the pattern of the previous proof. The property (2.13) is equivalent to $\overrightarrow{xD(x, Q)} = \overrightarrow{xQ}$, or $\overrightarrow{QD(x, Q} = 0$. Using again (2.10) we obtain $V(0, a) + A(0) = 0$, which is obviously true. Conversely, given a map $D : \mathcal{P}(M) \times \mathcal{P}_E(M) \to \mathcal{P}_E(M)$ which satisfies (2.11)-(2.13) we can derive from it a covariant differentiation ∇ and a section of $L(TM, E) \to M$, as follows:

$$\nabla(X, a) = \overrightarrow{QD(P, Q)} - \overrightarrow{xD(P, x)},\ A(X) = \overrightarrow{xD(P, x)}, \tag{2.14}$$

where
$$X = \overrightarrow{xP},\quad a = \overrightarrow{xQ}.$$

But the covariant differentiation ∇ does not define Γ, such that in our framework the map D does not determine a g.a.c. This happens when the dimension of M, as well as of E is finite (see [2]). It is also easy to prove, using (2.14), the following

Theorem 2.3. *A generalized affine connections on M is affine if and only if $D(P, x) = P$ for every $P \in \mathcal{P}(M)$.*

We obtain a g.a.c. on M when $E = TM$. Every section of $L(TM, TM) \to M$ is a tensor field of type (1,1). Therefore, we have:

Corollary 2.1. *Let M be a Banach manifold. There exists a one-to-one correspondence between the set of g.a.c. on M and the set of pairs consisting of a linear connection on M and a tensor field of type (1,1) on M.*

The g.a.c. ${}^a\Gamma$ which corresponds to (Γ, I), where I is the tensor of Kronecker, will be called affine connection.

Let $\widetilde{\Gamma}$ be a g.a.c. on the Banach manifold M and let be $K : TTM \to TM$ its connection map. The map D defined above, induces a covariant differentiation $\widetilde{\nabla} : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, which can be expressed as $\widetilde{\nabla}(X, Y) = K \circ TY(X)$, where TY is the tangent map to $Y : M \to TM$. Indeed, the local parts of $\widetilde{\nabla}$ and D are given by the right part of the equality

$$\widetilde{\nabla}_X Y|_\varphi = \partial Y_\varphi(X_\varphi) + \Gamma_\varphi(Y_\varphi, X_\varphi) + A(X_\varphi), \tag{2.15}$$

where X_φ, Y_φ are the local parts of X and Y, respectively and Γ_φ and A are the local components of g.a.c.

Let $c : (a, b) \subset R \to M$ be a smooth curve on M and let $Tc : (a, b) \times R \to TM$ be its tangent map. We denote by $\dot{c}$ the vector field on $c(a, b) \subset M$ given by $c(t) \to \dot{c}(t)$, where $t \in (a, b)$ and $\dot{c}(t) = Tc(t, 1)$. In a local chart (U, φ) with $U \cap c(a, b) \neq \emptyset$, the $\dot{c}(t)$ is given by

$$\dot{c}(t) = (c(t),\ \partial c_\varphi(t)(1)), \tag{2.16}$$

where $c_\varphi = \varphi \circ c$.

A curve c will be called a geodesic of the g.a.c. $\widetilde{\Gamma}$ if $\widetilde{\nabla}_{\dot{c}}\dot{c} = 0$. The local component c_φ of a geodesic of $\widetilde{\Gamma}$ satisfies the following differential equation

$$\partial^2 c_\varphi(t) + \Gamma_\varphi(\partial c_\varphi(t), \partial c_\varphi(t)) + A(\partial c_\varphi(t)) = 0. \tag{2.17}$$

From the theory of differential equations, it follows the local existence and the uniqueness of a geodesic with the initial conditions $c_\varphi(t_0) = c_0 \in M$ and $\partial c_\varphi(t_0)(1) = u_0 \in M$.

In the following, we shall prove that the well-known relationship between geodesics and sprays holds within the general context. A vector field S on TM, smooth on $TM - 0$, is said to be a spray on M if $T\pi \circ S = \mathrm{id}|_{TM}$. Let C be the canonical vector field on TM defined locally by $C(x, a) = (x, a, 0, a)$, $x \in U$, $a \in \mathbf{M}$.

Lemma 2.2. *A vector field S on TM, smooth on $TM - 0$, is a spray on M if and only if $J \circ S = C$, where J is the natural almost tangent structure on TM.*

Proof. A vector field S on TM can be written locally as follows

$$S(x, a) = (x, a, S_1(x, a),\ S_\varphi(x, a)),\ x \in U, a \in \mathbf{M}.$$

The condition $T\pi \circ S = \mathrm{id}|_{TM}$ implies $S_1(x, a) = a$, therefore $S(x, a) = (x, a, a, S_\varphi(x, a))$, where S_φ is smooth on $U \subset \mathbf{M} - 0$. It follows easily $J \circ S = C$, because $J(x, a, b, c) = (x, a, 0, b)$, $x \in U$, $a, b, c \in M$.

Conversely, given S as above, the condition $J \circ S = C$ implies $S_1(x, a) = a$, hence $T\pi \circ S = \mathrm{id}|_{TM}$.

Lemma 2.3. *Let K be the connection map of a nonlinear connection on M. There exists a unique spray on M such that $K \circ S = 0$, called geodesic spray.*

Proof. Locally, every spray S can be written as follows:

$$S(x, a) = (x, a, a, S_\varphi(x, a)).$$

We obtain the geodesics spray if we take $S(x,a) = \omega(x,a)\ a, x \in U,\ a \in \mathbf{M}$.

The geodesics of a nonlinear connection are the solutions of the following differential equation.

$$\partial^2 c_\varphi(t) + \omega(c_\varphi(t), \partial c_\varphi(t))\partial c_\varphi(t) = 0. \tag{2.18}$$

Theorem 2.4. *A curve $c : (a,b) \to M$ is a geodesic of a nonlinear connection N if and only if there exists an integral curve $\widetilde{c} : (a,b) \to TM$ of the geodesic spray S of N, such that $\pi \circ \widetilde{c} = c$.*

Proof. The curve $\widetilde{c}$ on TM is a integral curve of S if $\dot{\widetilde{c}} = S$, therefore in a local chart (U,φ) we have $\partial c_\varphi(t) = S_\varphi(\pi \circ \widetilde{c}_\varphi(t), \widetilde{c}_\varphi(t))$. Differentiating $\pi \circ \widetilde{c} = c$, we obtain $\widetilde{c}_\varphi(t) = \partial c_\varphi(t)$, therefore $\partial^2 c_\varphi(t) = S_\varphi(c_\varphi(t), \partial c_\varphi(t)) = -\omega(c_\varphi(t), \partial c_\varphi(t))\partial c_\varphi(t)$, or $\partial^2 c_\varphi(t) + \omega(c_\varphi(t), \partial c_\varphi(t))\partial c_\varphi(t) = 0$, i.e. c is a geodesic of N. Conversely, if c is a geodesic of N, then the curve $\widetilde{c}$ on TM given by $\widetilde{c}(t) = \dot{c}(t)$ is a integral curve of the geodesic spray of N and $\pi \circ \widetilde{c} = c$.

Remarks. As a corollary of the Theorem 2.4 one obtains again the local existence and uniqueness of a geodesic with given initial conditions. The geodesic spray of a g.a.c. is locally given by ${}^aS_\varphi(x,a) = (x,a,a,-\Gamma(x)(a,a) - A(x)a)$. Using (2.18) one obtains again the equation (2.17) for the geodesics of a g.a.c. Let us suppose that M has finite dimension. Then a curve c can be write as follows: $x^i = x^i(t),\ t \in (a,b),\ i = 1,2,...,m =dimM$ and the equation (2.17) becomes

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + A^i_j\frac{dx^j}{dt} = 0, \tag{2.19}$$

where Γ^i_{jk} are the Christoffel symbols of the linear connection associated to the g.a.c. and A^i_j are the components of a tensor of type (1.1) on M. The solutions of the equation (2.19), called holomorphically planar curves have been used to give some geometrical meanings in the geometry of complex manifolds [6].

When M is the space-time manifold of the general theory of relativity, the solutions of (2.19) are the trajectories of a charged particle moving in an electromagnetic field [1].

3 Flat generalized affine connections

Let $V : TE \to VE$ be the splitting map which defines a g.a.c. on $p : E \to M$. The map V can be viewed as a 1-form VE valued. On the other hand, the linear connection Γ defined by $\widetilde{\Gamma}$, induces a linear connection Γ_v on $VE \to E$.

Definition 3.1. The exterior differential dV of the 1-form $VE-$valued V, accounted using the linear connection Γ_v on $VE \to E$ will be called the curvature form of $\widetilde{\Gamma}$.

The local component of V, denoted also by $V : U \times \mathbf{E} \to L(\mathbf{M}, \mathbf{E}, \mathbf{E})$ is $V(x,a)(\lambda,b) = b + \Gamma(x)(a,\lambda) + A(x)\lambda,\ x \in U,\ \lambda \in M,\ a,b \in \mathbf{E}$. After a calculus rather long but not difficult, one obtains the following local expression for the curvature of form $\widetilde{\Gamma}$:

$$\begin{aligned} dV(z,a)((\lambda,b),(\mu,c)) &= R(x)(\lambda,\mu) + \partial A(x)(\lambda,\mu) - \\ &-\partial A(x)(\mu,\lambda) + \Gamma(x)(A(x)\mu,\lambda) - \Gamma(x)(A(x)\lambda\mu), \end{aligned} \tag{3.1}$$

$x \in U$, $\lambda, \mu \in \mathbf{M}$, $a, b, c \in \mathbf{E}$, where $R(x)$ is the local component of the curvature tensor of Γ.

From (3.1) it follows that dV vanishes when it is applied to a vertical vector field ($\lambda = 0$ or $\mu = 0$), therefore dV is an horizontal 2-form i.e.

$$dV(\mathbf{A}, \mathbf{B}) = dV(h\mathbf{A}, h\mathbf{B}) \tag{3.2}$$

holds for every vector fields $\mathbf{A}, \mathbf{B}$ on E. Using $dV(\mathbf{A}, \mathbf{B}) =^V \nabla_{\mathbf{A}} V\mathbf{B} -^V \nabla_{\mathbf{B}} V\mathbf{A} - V[\mathbf{A}, \mathbf{B}]$, where ${}^V\nabla$ is the covariant differentiation associated to Γ_v and (3.2), one obtains.

$$dV(\mathbf{A}, \mathbf{B}) = V[h\mathbf{A}, h\mathbf{B}] \text{ (the structure equation of } \overline{\Gamma}). \tag{3.3}$$

Definition 3.2. A g.a.c. will be called flat if its horizontal distribution is involutive i.e. the bracket of two horizontal vector fields is again a horizontal vector field.

From the structure equation (3.3) it follows

Theorem 3.1. *The g.a.c.* $\widetilde{\Gamma}$ *is flat if and only if* $dV = 0$.

Taking $dV = 0$ and $a = 0$ in (5.1) one obtains

$$\partial A(x)(\lambda, \mu) - \partial A(x)(\mu, \lambda) + \Gamma(x)(A(x)\mu, \lambda) - \Gamma(x)(A(x)\lambda, \mu) = 0. \tag{3.4}$$

Taking again $dV = 0$ in (3.1) and using (3.4) one obtains

$$R(x)(\lambda, \mu)a = 0. \tag{3.5}$$

Conversely, if (3.4) and (3.5) hold, then $dV = 0$, therefore we have proved the following

Theorem 3.2. *The g.a.c.* $\widetilde{\Gamma} = (\Gamma, A)$ *is flat if and only if the curvature tensor* R *of* Γ *vanishes identically and (3.4) holds.*

When $p = \pi : TM \to M$, the conditions (3.4) is equivalent to the vanishing of the following tensor of type $(1, 2)$ on M

$${}^aT(X, Y) = \nabla_X A(Y) - \nabla_Y A(X) - A[X, Y], \tag{3.6}$$

where X, Y are vector fields on M, which will be called the torsion tensor of g.a.c. $\widetilde{\Gamma}$. The Theorem 3.2 has the following

Corollary 3.2. *A g.a.c.* $\widetilde{\Gamma} = (\Gamma, A)$ *on the manifold* M *is flat if and only if* $R = 0$ *and* ${}^aT = 0$.

When $A = I$ (the Kronecker tensor), ${}^aT$ becomes the well-known torsion tensor of an a affine connection, therefore we have

Corollary 3.3. *An affine connection on a Banach manifold M is flat if and only if its curvature tensor and torsion tensor are both identically zero.*

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University of Iasi
Faculty of Mathematics

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SOME EXISTENCE THEOREMS IN FINSLER GEOMETRY

BY

M. ANASTASIEI

Let M be a differentiable manifolds of class C^k $(k \geq 3)$ and let $p : TM \to M$ be the tangent bundle to it. A positive real valued function $L : TM \to R^+$, with properties:

1) L is differentiable of class C^{k-1} on $TM \setminus O$ and continuous on the image of the null-section of p,

2) Its local representation in every chart $(p^{-1}(U_i), x^i, y^i)$ on TM induced by the chart (U, x^i) on M, denoted by $L(x^i, y^i)$ or for brevity by $L(x, y)$ is (1) $p-$homogeneous i.e. $L(x, sy) = sL(x, y)$ for every $s > 0$,

3) The matrix $(g_{ij}(x, y)) = \left(\dfrac{\partial^2(L^2(x, y)/2)}{\partial y^i \partial y^j}\right)$ is invertible and the quadratic form associated to it is positive definite,

is called *fundamental Finsler function* and the pair (M, L) is called a Finsler space.

The matrix $(g_{ij}(x, y))$ changes, when the local chart changes, as the components of a tensor of type $(0, 2)$ on M but it depends on direction (given by y^i) so $g_{ij}(x, y)$ defines a Finsler tensor field of type $(0, 2)$ (see [1] for a general definition of Finsler geometric objects). New fields of Finsler geometric objects (tensors, connections) can be derived from L. Obviously, their existence is assured by the existence of L.

As it was pointed out in [3, p. 81], the conditions imposed on L are too restrictive. It was an idea of R. Miron to eliminate the function L and to define a Finsler space as a pair (M, g), where g is a symmetrical Finsler tensor field of type $(0, 2)$, nondegenerate, positive definite or not. He called such a g a metrical Finsler structure on M. The fields of Finsler geometric objects can be defined independent on L or g. So the problem of their existence, in particular that of the existence of L and g is quite natural. The aim of this paper is to prove the global existence of Finsler tensor fields, of linear Finsler connections and of metrical Finsler structures in the hypothesis M paracompact, modeled by a separable Hilbert space. In a second section we make more explicit a proof due to S. Kashiwabara [2] of the global existence of a fundamental function L, in the hypothesis M finite dimensional and paracompact. Some comments regarding the existence of L when M is infinite dimensional are made.

1 The global existence of Finsler geometric objects

Let us state again the hypothesis on M in this section: differentiable of class C^k ($k \geq 3$), modeled by a separable Hilbert space H and paracompact i.e it is separate Hausdorff and every open covering of it admits a locally finite refinement. We recall that a C^k-partition of unity on a manifold X is an indexed family of C^k real valued functions $\{f_j\}_{j\in J}$ on X such that

1) $f_j \geq 0$,
2) $\{\text{supp}(f_j)\}_{j\in J}$ is locally finite and
3) $\sum_j f_j(x) = 1$, $x \in X$.

The partition of unity $\{f_j\}_{j\in J}$ is said to be subordinate to the covering $\{U_i\}_{i\in I}$ of X if $\{\text{supp}(f_j)\}_{j\in J}$ refines $\{U_i\}_{i\in I}$.

As it was proved in [6, p.57-60], every separable Hilbert space admits C^k-partition of unity and a necessary and sufficient condition that a C^k-manifold X admit C^k-partition of unity is that X be paracompact and that each of its tangent spaces admit C^k-partition of unity. Consequently, our manifold M admits C^k-partition of unity.

Let $A = \{(U_i, \varphi_i)\}_{i\in I}$ be an atlas on M. Suppose that A is maximal i.e. it contains all charts compatible with it. Then $\{U_i\}$ is a basis for the topology of M. Let x be a point of M and let (U_i, φ_i), (U_j, φ_j) be two local charts around x. The triads (U_i, φ_i, u) and (U_j, φ_j, v), where $u, v \in H$, are called equivalent if $D_{\varphi_i}(x)(\varphi_j \circ \varphi_i^{-1})(u) = v$, where D means Fréchet differentiation. This is indeed an equivalence relation on the set of such a triad s and the class of equivalence $[(U_i, \varphi_i, u)]$ is called vector tangent to M in x. Thus every chart (U_i, φ_i) defines a map $\theta_{i,x} : T_xM \to H$, $\theta_{i,x}([U_i, \varphi_i, u)] = u$. We set $TM = \cup_{x\in M} T_xM$ and $p : TM \to M$ projects T_xM on x. The topology and the differentiable structure of TM are induced by those of M. As a basis for the topology of TM is taken $(p^{-1}(U_i))_{i\in I}$ and p becomes a continuous map. One verifies easily that $(p^{-1}(U_i), h_i)$, where $h_i : p^{-1}(U_i) \to H \times H$, $h_i(z) = (\varphi_i(p(z)), \theta_{i,p(z)}(z)$ is a $C^{k-1}-$atlas (not maximal) on TM. Usually the differentiable structure defined by this atlas is considered. The map p becomes a $C^{k-1}-$submersion.

Theorem 1.1. *The manifold TM is paracompact.*

Proof. Let $z_1, z_2 \in TM$ and let us denote $x_1 = p(z_1)$, $z_2 = p(z_2)$. There exist open sets D_1 and D_2 such that $x_1 \in D_1$, $x \in D_2$ and $D_1 \cap D_2 = \emptyset$. Putting $D_1 = \cup_{j\in I_1} U_j$ and $D_2 = \cup_{j\in I_2} U_j$, it follows that there exist $j_1 \in I_1$, $j_2 \in I_2$ such that $x_1 \in U_{j1}$, $x_2 \in U_{j2}$ and $U_{j1} \cap U_{j2} = \emptyset$. Then $p^{-1}(U_{j1}) \cap p^{-1}(U_{j2}) = \emptyset$ and $z_1 \in p^{-1}(U_{j1})$, $z_2 \in p^{-1}(U_{j2})$, therefore TM is separate Hausdorff.

Let $\{D_j\}_{j\in J}$ be an open covering of TM. We may write $D_j = \cup_{i\in I} p^{-1}(U_i)$. The open covering $\{U_i\}_{i\in I}$ admits an open locally finite refinement $\{V_k\}_{k\in K}$ i.e. for every $k \in K$ there exists $i(k) \in I$ such that $V_k \subset U_{i(k)}$. It follows $p^{-1}(V_k) \subset p^{-1}(U_{i(k)})$ and obviously there exists $D_{j(k)} \supset p^{-1}(U_{i(k)})$, therefore $(p^{-1}(V_k))_{k\in K}$ is an open refinement of the covering $\{D_j\}$. We prove that it is locally finite. If $z \in TM$ and $x = p(z)$, there exists an open neighborhood U

of x which intersects only a finite number of V's say $V_1, ..., V_n$. It follows by reductio ad absurdum that $p^{-1}(U)$ intersects only $p^{-1}(V_1), ..., p^{-1}(V_n)$. The proof is complete.

Corollary 1.1. *Let M be a paracompact manifold modeled by a separable Hilbert space H. Then the manifold TM admits C^{k-1}−partition of unity.*

Proof. The space $H \times H$ being the product of two separable Hilbert spaces is itself a separable Hilbert space. The manifold TM being paracompact, the proof follows via the above-mentioned theorems.

Corollary 1.2. *Let M be a finite dimensional manifold of class C^k paracompact. Then TM is a paracompact manifold of class C^{k-1} and it admits a C^{k-1}−partition of unity.*

Proof. Obvious.

A partition of unity for TM can be obtained from a partition of unity for M as follows:

Theorem 1.2. *Let $\{f_j\}_{j\in J}$ be a C^k−partition of unity on M subordinated to the covering $\{U_i\}_{i\in I}$. Then $\{f_j^V = f_i \circ p\}_{jeJ}$ is a C^{k-1}−partition of unity on TM which is subordinated to the covering $\{p^{-1}(U_i)\}_{i\in I}$.*

Proof. Obviously, $f_j^V \geq 0$ for every $j \in J$. Then $\text{carr} f_j^V = \{z \in TM | f_j(p(z)) \neq \emptyset\} \subset p^{-1}(\text{supp} f_j)$, therefore $\text{supp} f_j^V \subset p^{-1}(\text{supp} f_j) \subset p^{-1}(U_{i(j)})$ because $\text{supp} f_j$ is closed. Since $\{\text{supp} f\}_{j\in J}$ is locally finite, so is $\{\text{supp} f_j^v\}_{j\in J}$. The equalities $\sum f_i^V(z) = \sum f_i(p(z)) = 1$ end the proof.

The fields of Finsler geometric objects can be obtained as cross-sections of a convenient fibre bundle over TM. We recall briefly the construction of that bundle $FO^k \to TM$ (see [5]). Let us denote by $L_k(H)$ the set of k−jets of source $0 \in H$ of the local diffeomorphism of H which preserves $0 \in H$. The composition of k−jets gives a group structure on $L_k(H)$. The set $P^k(M)$ of all k−jets of source $0 \in H$ of the local diffeomorphisms of H to M can be structured as a principal fibre bundle over M with structural group $L_k(H)$. The pull-back by p of this bundle will be denoted by $F^k(M) \to TM$ (this is the Finsler bundle of order k).

A pair (F, m), where F is a manifold and m a differentiable action of $L_k(H)$ on F, is called a manifold of geometric objects. Usually F is taken a linear space or an open subset of a linear space. The fibre bundle associated to $F^k(M)$ of type fiber (F, m) is denoted by $FO^k \to TM$ and is called the bundle of Finsler geometric objects. Its cross-sections are called fields of Finsler geometric objects on M. If F is a linear space and m is a linear action on F, the cross-sections of the bundle of Finsler geometric objects are called fields of linear Finsler geometric objects.

Now to state a result proved in [6, p. 62], some definitions are necessary. A subset $C \subseteq E$, where E is the total space of a C^k−bundle $q : E \to M$, is said to be convex if for each $x \in M$, $C_x = C \cap E_x$ is a nonvoid and convex set. (Here E_x denotes the fiber in x.) One says C admits local C^k−sections if given $c_0 \in C$ with $q(c_0) = x_0$, there is an open neighborhood U of x_0 and a C^k−section s over U with $s(x_0) = c_0$ and $s(U) \subseteq C$.

Theorem 1.3. [6] *Let $q : E \to M$ be a C^k-bundle and let C be a convex subset of E which admits local C^k-sections. There exists a C^k-section S of q over M such that $S(x) \in C$ for every $x \in M$.*

The bundle $F0^k \to TM$ associated to $P^k(M) \to TM$ with F a linear space and m a linear action admits local C^k-sections, because it is a locally trivial vector bundle (every bundle chart of it defines a local C^k-section). Using the Theorem 1.3 one obtains

Theorem 1.4. *Let M be a paracompact manifold modeled by a separable Hilbert space. There exist global fields of linear Finsler geometric objects on M i.e. cross-sections over TM of vector bundle $FO^k \to TM$.*

The Finsler tensor fields of type $(0, r)$ or $(1, r)$, $r \geq 1$ can be considered as linear Finsler fields of geometric objects by adjusting in an obvious manner a definition from finite dimensional case (see [1]). So, we have

Corollary 1.5. *Under the hypothesis of the above theorem, there exist globally on M, Finsler vector fields and Finsler tensor fields of type $(0, r)$ and $(1, r)$.*

Corollary 1.6. *Let M be a paracompact finite dimensional manifold. Then there exists global Finsler tensor fields of any type on M.*

Let us take $F = L_2(H, H)$ i.e. the linear space of linear maps $H \times H \to H$ and let the action m be denoted by m_c and defined by $m_c : L_2(H) \times L_2(H, H) \to L_2(H, H)$, $m_c(h, K) = AK(A^{-1}, A^{-1}) - A_1(A^{-1}, A^{-1})$ if $h = (A, A_1) \in L_2(H)$. With this choice of F and m, the cross-sections of $p_c : FO^2 \to TM$ are called linear Finsler connections on M. The fiber $p_c^{-1}(z)$, $z \in TM$, is not a linear space although its elements can be added and multiplied by reals, because m_c is not linear. However it is a convex set because if $a + b = 1$, $a, b \in R$, $m_c(h, aK_1 + bK_2) = am_c(h, K_1) + bm_c(h, K_2)$ holds good. The bundle p_c is locally trivial hence it admits local C^k-sections. The Theorem 1.3 applies and leads to

Theorem 1.7. *On every paracompact manifold modeled by a separable Hilbert space there exist global Finsler linear connections.*

Corollary 1.8. *On every paracompact finite dimensional manifold there exist global Finsler linear connections.*

Let us now take $F = L_2^s(H, R)$, the linear space of real valued bilinear and symmetrical maps on H and $m : GL(H) \times L_2^s(H, R) \to L_2^s(H, R)$ given by $m(A, g) = g(A^{-1}, A^{-1})$, where A belongs to the general linear group $GL(H)$ of H. The cross-sections of the bundle $p_1 : FO^1 \to TM$ obtained with such a choice of F and m are symmetric Finsler tensor fields of type $(0, 2)$. Every $g \in L_2^s(H, R)$ defines a linear operator $\widetilde{g} : H \to H^*$. If g is invertible, g is called nondegenerate. The set $\mathcal{R}(H, R)$ of all nondegenerate symmetrical bilinear maps on H is an open and convex subset of $L_2^s(H, R)$. Obviously, m leaves invariant the subset $\mathcal{R}(H, R)$. By applying the general construction sketched above taking $\mathcal{R}(H, R)$ as F, one obtains a fibre bundle over TM whose total space $\mathcal{R}FO^1$ is a subset of FO^1 which is clearly open and convex. Being open it admits local C^k-sections, therefore by applying the Theorem 1.3 one obtains

Theorem 1.9. *On every paracompact manifold modeled by a separable Hilbert space, there exist global metrical Finsler structures i.e. cross-sections $g : TM \to FO^1$ such that $g(TM) \subset \mathcal{R}FO^1$.*

Corollary 1.10. *On every paracompact finite dimensional manifold there exist global metrical Finsler structures.*

When F is a linear space and m a linear action, the bundle $FO^k \to TM$ can be identified (is isomorphic) to a vector bundle obtained from the vertical subbundle V of $TTM \to TM$ by algebraic operations. So, the Finsler vector fields appear as sections of $V \to TM$, the Finsler tensor fields of type $(0, r)$ appear as sections of $L(V, ..., V; R) \to TM$, the Finsler tensor fields of type $(1, r)$ are sections of $L(V, ..., V; V) \to TM$ and so on.

A Finsler almost product structure on M is a section $P : TM \to L(V, V)$ which satisfies $P(z) \circ P(z) = I$, where I is identity map on fiber V_z. The global existence of such a structure on M is a consequence of the following fact: if V' is a subbundle of V, there exists a subbundle V'' of V such that $V' \oplus V'' = V$. Its proof is standard, using a partition of unity on TM. So, if $s_z = s'_z + s''_z$ where $s'_z \in V'_z$ and $s''_z \in V''_z$ we may define $P(s'_z) = s'_z$, $P(s''_z) = s''_z$ and it follows $P \circ P = I$.

2 The global existence of a fundamental Finsler function

Let us suppose that M is a separate Hausdorff, finite dimensional manifold, satisfying the second axiom of countability. It follows that M is paracompact, therefore it admits C^k-partition of unity. We shall prove the existence of a real valued function L on TM verifying the conditions to be a fundamental Finsler function. Firstly, we prove the following

Lemma 2.1. *Let n be the dimension of M. There exists a continuous function $f : R^n \to R^+$ which is*

a) 1 $(p)-$homogeneous,

b) differentiable at least of class C^3 on the complement of the origin and the quadratic form $\dfrac{\partial^2(f^2(y)/2)}{\partial y^i \partial y^j} z^i z^j$ is positive definite for all values of $z^i \neq 0$, where $y = (y^1, ..., y^n)$, $z = (z_1, ..., z^n)$ belong to R^n. Furthermore, f is a norm on R^n.

Proof. Let $h : R^n \to R$ be a continuous function $1(p)-$homogeneous, differentiable at lest of class C^3 on the complement of the origin and $h(0) = 0$. Such a function always exists. For instance we may take $h(y) = \left(\sum_{i=1}^{n} (y^i)^p \right)^{1/p}$ with $p \geq 1$, $p \neq 2$. Let us put $f(y) = \left(\sum_{i=1}^{n} (y^i) + \varepsilon h^2(y) \right)^{1/2}$, where ε is a positive real number. Obviously, f is $1(p)-$homogeneous and differentiable at least of class C^3 on the complement of the origin. The matrix

$A = \left(\dfrac{\partial^2(f^2/2)}{\partial y^i \partial y^j}\right)$ is given as follows: $A = I + B$, where I is the unity matrix and $B = \left(\dfrac{\partial h}{\partial y^i} \cdot \dfrac{\partial h}{\partial y^j} + h(y)\dfrac{\partial^2 h}{\partial y^i \partial y^j}\right)$. Choosing $\varepsilon < 1/\|B\|$, where $\|B\|$ means a norm on the space of matrices, A becomes an invertible matrix and furthermore, the quadratic form associated to A becomes positive definite. It is obvious that $f(y) \geq 0$, the equality sign occurring only if $y = 0$. The condition $f(sy) = |s|f(y)$ is clearly satisfied. A proof of the inequality $f(x+y) \leq f(x) + f(y)$, $x, y \in R^n$ can be performed using a method of H. Rund [7, p. 18–20].

Theorem 2.1. *Let M be a finite dimensional paracompact manifold. There exists a fundamental Finsler function on TM.*

Proof. Let $(a_i)_{i\in I}$ be a C^k-partition of unity ($k \geq 1$) on M subordinate to a covering $\{U_i\}_{i\in I}$ of M, where U_i, is the domain of a bundle chart $\varphi_i : p^{-1}(U_i) \to U_i \times R^n$. Define $L_i : U_i \times R^n \to R$ by $L_i(p,u) = f(u)$, where f is given by the Lemma 2.1. The function L defined by $L(v_p) = \sum_i a_i(p)L_i(\varphi_i(v_p))$, $v_p \in TM$, satisfies all requirements to be a fundamental Finsler function.

In his lectures given at Brandeis University in 1965, R. S. Palais has considered what he called Finsler structures on a Banach bundle, in particular on a Banach manifold. Following his definition, a Finsler structure on M is a function $L : TM \to R^+$ such that for every $p_0 \in M$ there exists a bundle chart $\varphi : p^{-1}(U) \to U \times H$ such that $L \circ \varphi^{-1}$ verifies

1) $(L \circ \varphi^{-1})(p_0) : H \to R^+$ is an admissible norm on H,

2) There exists a neighborhood $U_0 \subset U$ of p_0 such that $(L \circ \varphi^{-1})(p)$ be an equivalent norm to $(L \circ \varphi^{-1})(p_0)$ for every $p \in U_0$.

This notion is less restrictive then the usual notion of Finsler structure in finite dimensions, where the second condition becomes trivial. The existence of a function L satisfying 1) and 2) was proved by R. S. Palais by means of a partition of unity on M. The basic tool in his proof was the so-called flat Finsler structure given by the map $N : M \times H \to R^+$, $N(p,u) = \|u\|$, for every $p \in M$ and $u \in H$, where $\|\cdot\|$ is the norm induced by the inner product of H. Such a Finsler structure satisfies a third condition

3) $D_u^2(L \circ \varphi^{-1})^2/2$ is for every $u \in H$ an isomorphism of H to H^*, but it is essential a Riemannian one. Here D^2 means Fréchet differentiation of the second order.

When the norm on H is not Hilbertian i.e. H is a separable Banach space which admits a partition of unity, the procedure of R. S. Palais leads to a proper Finsler structure, but if the condition 3) is imposed, it becomes a Riemannian one. This happens since the condition 3) implies that the norm of H is equivalent to a norm induced by an inner product of H. We give in the following a proof of this assertion. Let p_0 be a fixed point of M and let us put $g = (L \circ \varphi^{-1})^2/2 = N^2/2 = \|\cdot\|^2/2$ and $T = D_u^2 g$. The map T can be viewed as a continuous and symmetrical bilinear form on H. Differentiating g, one obtains $D_u^2 g(v,v) = (D_u N(v))^2 + \|u\|^2 D_u^2 N(v,v) \geq 0$ for every $v \in H$ since $D_u^2\|\cdot\|(v,v) \geq 0$ (see [7]). Therefore T is also a positive bilinear form on H, hence the inequality of Cauchy–Schwarz $|T(u,v) \leq p(u)p(v)$, where $p(u) = T(u,u)^{1/2}$, holds. If $p(v) = 0$ it follows $T(u,v) = 0$ for every $u \in H$

or $(T(v))(u) = 0$ for every $u \in H$ and $v = 0$ because T is an isomorphism. So, T is an inner product on H. The inequality $p(v) \leq \|T\|^{1/2}\|v\|$ is obvious. Let us take $S = \{v \in H | p(v) \leq 1\}$. Then the inequality $|T(u)(v)| \leq p(v)$ holds for every $u \in S$. It follows there exists $c > 0$ such that $\|T(u)\| \leq c$ for every $u \in S$. We have $\|u\| = \|(T^{-1} \circ T)(u)\| \leq \|T^{-1}\| \, \|T(u)\| \leq \|T^{-1}\|c$ for every $u \in S$. So, $\|u\| \leq c\|T^{-1}\|p(u)$ for every $u \in H$. Therefore the norms $\|\cdot\|$ and p are equivalent.

The above considerations show that in the framework of Banach manifolds the definition of Finsler structures given by R. S. Palais is the most convenient.

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Faculty of Mathematics
University of Iasi

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL. I. CUZA" IAŞI
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VECTOR BUNDLES. EINSTEIN EQUATIONS

BY

M. ANASTASIEI

In the last years a Finslerian theory of relativity was built from various standpoints [2], [4], [7]. Recently, R. Miron has completed a more generalized version of this theory, which was called a Lagrangian theory of relativity in [6]. Some physical aspects of this theory were considered by S. Ikeda in [3]. His considerations show that the geometry of the total space of a vector bundle is useful from a physical viewpoint.

In this paper the Einstein equations and the conservation law on the total space of a vector bundle are written. If the vector bundle is just the tangent bundle to the base manifold we recover the Einstein equations established by R. Miron in [6] as well as a new kind of Einstein's equations whose physical meaning remains to be found. If the vector bundle has 1-dimensional fibres, we obtain a geometrical framework for an unitary projective theory.

The author is indebted to Prof. Radu Miron who suggested him the subject of this work.

1 Vector bundles

Let $\xi = (E, p, M)$, $p : E \to M$, be a vector bundle of paracompact base M and finite dimensional type fibre $\mathbf{F}$. We set $n = \dim M$ and $m = \dim \mathbf{F}$. Let us denote, by (x^i, y^a) the local coordinates on $p^{-l}(U) \subset E$, where $U \subset M$. In what follows we use $i, j, k, h ... = 1, 2, ..., n$ and $a, b, c, ... = 1, 2, ..., m$.

The law of transformation of the local coordinates is the following:

$$x^i = x^i(x^1, ..., x^n),\ y^{a'} = S_a^{a'}(x^1, ..., x^n)y^a. \tag{1.1}$$

If the vector bundle is endowed with a nonlinear connection, then, for every $u \in E$, we have $T_uE = H_uE \oplus V_uE$, where V_uE is the vertical part and H_uE is the horizontal part. A basis of T_uE adapted to this decomposition is (δ_i, ∂_a), where $\delta_i = \partial_i - N_i^a(x, y)\partial_a$. Here $(N_i^a(x, y))$ are the local coefficients of the nonlinear connection and ∂_i and ∂_a stand for $\partial/\partial x^i$ and $\partial/\partial y^a$, respectively. The basis dual to it is $(dx^i, \delta y^a)$, where $\delta y^a = dx^a + N_i^a dx^i$.

Definition 1.1. A linear connection D on the manifold E is said to be a $d-$connection if it preserves by parallel displacement the horizontal distribution $u \to H_uE$ and vertical distribution $u \to V_uE$.

If we set:

$$
(1.2)\qquad \begin{cases} D_{\delta_k}\delta_j = F^i_{jk}(x,y)\delta_i,\ \ D_{\delta_k}\partial_b = L^a_{bk}(x,y)\partial_a, \\ \\ D_{\partial_a}\delta_j = M^i_{ja}(x,y)\delta_i,\ \ D_{\partial_c}\partial_b = C^a_{bc}(x,y)\partial_a, \end{cases}
$$

then $F^i_{jk}(x,y)$ and $L^a_{bk}(x,y)$ change like the local coefficients of a connection on M, respectively on ξ, and $M^i_{ja}(x,y)$, $C^a_{bc}(x,y)$ are tensor fields on E. A $d-$connection is completely determined by $F\Gamma = (F^i_{jk}, L^a_{bk}, M^i_{ja}, C^a_{bc})$. (See also [5]).

There exist $d-$connections on E. For instance, if $F^i_{jk}(x)$ are the local coefficients of a linear connection on M (there exists such a connection because M is paracompact), then $(F^i_{jk}(x), \partial_b N^a_j, 0, 0)$ is a $d-$connection on E.

A pair of linear connections on M and ξ defines a $d-$connection on E. Indeed, if $L^a_{bk}(x)$ are the local coefficients of a linear connection on ξ, then $N^a_i(x,y) = L^a_{bk}y^b$ are the local coefficients of a nonlinear connection on ξ and $(F^i_{jk}(x), L^a_{bk}(x), 0, 0)$ is a $d-$connection.

A $d-$connection $F\Gamma$ is called a Berwald connection if

$$
L^a_{bk} = \partial_b N^a_k(x,y),\ \ M^i_{ja}(x,y) = 0.
$$

The $d-$connections showed above are Berwald connections. We shall denote by $_|$ and $|$ the $h-$ and $v-$covariant derivative, respectively, associated to the $d-$connection D.

The Ricci identities introduce five torsions:

$$
(1.3)\qquad \begin{cases} T^i_{jk} = F^i_{jk} - F^i_{jk},\ \ R^a_{jk} = \delta_k N^a_j - \delta_j N^a_h,\ \ \overset{1}{P}{}^a_{jb} = \partial_b N^a_j - L^a_{bj}, \\ \\ \overset{2}{P}{}^i_{jb} = M^i_{jb},\ \ S^a_{bc} = C^a_{bc} - C^a_{cb}, \end{cases}
$$

and six curvatures:

$$
(1.4)\qquad \begin{cases} R^i_{jkh} = \delta_h F^i_{jk} + F^l_{jk}F^l_{ih} - k|h + M^i_{ja}R^a_{kh}, \\ \\ R^a_{bkh} = \delta_h L^a_{bk} + L^c_{bk}L^a_{ch} - k|h + C^a_{bc}R^c_{kh}, \\ \\ \overset{1}{P}{}^a_{bkc} = \partial_c L^a_{bk} - C^a_{bc|k} + C^a_{bd}P^s_{kc}, \\ \\ \overset{2}{P}{}^i_{jkc} = \partial_c F^i_{jk} - M^i_{jc|k} + M^i_{jb}\overset{1}{P}{}^b_{kc}, \\ \\ M^i_{jbc} = \partial_c M^i_{jb} + M^h_{jb}M^i_{hc} - b|c, \\ \\ S^a_{bcd} = \partial_d C^a_{bc} + C^e_{bc}C^a_{ed} - c|d, \end{cases}
$$

for a $d-$connection $F\Gamma$. Here and in the following $-k|h$ means the substraction of the previous terms after having changed the indices one to another one.

2 Metrical structures on E. Metrical $d-$connections

A metrical structure on E is a tensor field G on E of type $(0,2)$, symmetric and nondegenerate. If such a metrical structure G is given, then there exists a canonical nonlinear connection on ξ defined by the orthogonal distribution to the vertical distribution with respect to G. In what follows we shall refer only to this nonlinear connection. It is obvious that, with respect to the adapted frame to this nonlinear connection, G can be written as follows:

$$G = g_{ij}(x,y)dx \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b. \tag{2.1}$$

Definition 2.1. A $d-$connection on E is said to be metrical if

$$g_{ij|k} = 0,\ \ g_{ij}|_a = 0,\ \ h_{ab|k} = 0,\ \ h_{ab}|_c = 0, \tag{2.2}$$

hold.

There exist metrical $d-$connections. Indeed, if $\overset{\circ}{F\Gamma} = (\overset{\circ}{F}{}^i_{jk}, \overset{\circ}{L}{}^a_{bk}, \overset{\circ}{M}{}^i_{ja}, \overset{\circ}{C}{}^a_{bc})$ is any $d-$connection on E, then the $d-$connection whose local coefficients are given below is metrical.

$$\begin{cases} F^i_{jk} = \overset{\circ}{F}{}^i_{jk} + \dfrac{1}{2} g^{ih} g_{hk\overset{\circ}{_|}j} \\[2ex] L^a_{bj} = \overset{\circ}{L}{}^a_{bj} + \dfrac{1}{2} h^{ac} h g_{cb\overset{\circ}{_|}a} \\[2ex] M^i_{jb} = \overset{\circ}{M}{}^i_{jb} + \dfrac{1}{2} g^{ih} g_{hi\overset{\circ}{|}b} \\[2ex] C^a_{bc} = \overset{\circ}{C}{}^a_{bc} + \dfrac{1}{2} h^{ad} g_{db\overset{\circ}{|}c}, \end{cases} \tag{2.3}$$

where $\overset{\circ}{_|}$ and $\overset{\circ}{|}$ denotes the $h-$ and $v-$covariant derivative, respectively, associated to $\overset{\circ}{F\Gamma}$.

The formulas (2.3) can be thought of as a process of metrization of any $d-$connection. This process will be called Kawaguchi metrization.

We say that a $d-$connection is $h-v-$metrical with respect to G given by (2.1), if $g_{ij|k} = 0$ and $h_{ab}|_c = 0$. We remark that there exist $h-v-$metrical connections which are not metrical. Indeed, it is easy to check that the

following Berwald connection

$$(2.4)\qquad \begin{cases} \widetilde{F}^i_{jk} = \dfrac{1}{2} g^{ih}(\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}) \\ \\ \widetilde{L}^a_{bj} = \partial_b N^a_j \\ \\ \widetilde{M}^i_{jb} = 0 \\ \\ \widetilde{C}^a_{bc} = \dfrac{1}{2} h^{ad}(\partial_b h_{dc} + \partial_c h_{db} - \partial_d h_{bc}) \end{cases}$$

is $h-v-$metrical but it is not metrical.

Theorem 2.1. *If two skew-symmetrical tensor fields T^i_{jk} and S^a_{bc} are given, then there exists an unique Berwald connection which is $h-v-$metric and has $h(hh)-$ and $v(vv)-$torsions the tensor fields T^i_{jk} and S^a_{bc}, respectively. Its local coefficients are as follows:*

$$(2.5)\qquad \begin{cases} \widehat{F}^i_{jk} = \widetilde{F}^i_{jk} + \dfrac{1}{2} g^{ih}(g_{hr} T^r_{jk} - g_{jr} T^r_{hk} + g_{kr} T^r_{jh}) \\ \\ \widehat{L}^a_{bk} = \partial_b N^a_k \\ \\ \widehat{M}^i_{ja} = 0 \\ \\ \widehat{C}^a_{bc} = \widetilde{C}^a_{bc} + \dfrac{1}{2} h^{ad}(h_{de} S^e_{bc} - h_{be} S^e_{dc} + h_{ce} S^e_{bd}). \end{cases}$$

Proof. All Berwald connections have the form $(F^i_{jk} + \tau^i_{jk},\ \partial_b N^a_k, 0, C^a_{bc} + \widetilde{\tau}^a_{bc})$, where τ^i_{jk} and $\widetilde{\tau}^a_{bc}$ are arbitrary tensor fields. Imposing that such a connection be $h-v-$metrical and its $h(hh)-$ and $v(vv)-$torsions to be just T^i_{jk} and S^a_{bc}, respectively, one obtains that τ^i_{jk} and $\widetilde{\tau}^a_{bc}$ are uniquely determined and they have the expressions from (2.5), q.e.d.

Theorem 2.2. *There exists an unique metrical d-connection with $h(hh)-$ and $v(vv)-$torsions T^i_{jk} and S^a_{bc} prescribed obtained by Kawaguchi metrization of a $h-v-$metrical Berwald connection. Its local coefficients are as follows:*

$$(2.6)\qquad \begin{cases} F^i_{jk} = \widetilde{F}^i_{jk} \\ \\ L^a_{bj} = \partial_b N^a_k + \dfrac{1}{2} h^{ac}[\delta_k g_{bc} - (\partial_b N^d_k) k_{dc} - (\partial_c N^d_k) h_{db}] \\ \\ M^i_{jb} = \dfrac{1}{2} g^{ik} \partial_b g_{jk} \\ \\ C^a_{bc} = \widetilde{C}^a_{bc}. \end{cases}$$

Proof. By the Kawaguchi metrization of the unique Berwald $h-v-$metrical connection given by (2.5) one obtains (2.6), q.e.d.

3 Einstein equations on E

Let E be the total space of the vector bundle (E,p,M). Suppose that E is furnished with a metrical structure G and denote by D the metrical $d-$connection having $h(hh)-$and $v(vv)-$torsions prescribed, given locally by (2.6).

We associate to D the following Einstein equation

$$\text{Ric}(D) - (1/2)\mathbf{R}G = \varkappa\mathbf{T}, \tag{3.1}$$

where Ric(D) and $\mathbf{R}$ denote the Ricci tensor and the scalar curvature of D, respectively, $\varkappa$ is a constant and $\mathbf{T}$ is a tensor field of type $(0,2)$ called the energy-momentum tensor.

Remark 3.1. The tensor field from the left hand of the eq. (3.1) is not symmetric nor free divergence since D has torsion.

To express (3.1) by using the curvature of the $d-$connection D, let us put $X_a = \{\delta_i, \partial_a\}$. Then we have:

$$D_{X_\gamma}X_\beta = \Gamma^\alpha_{\beta\gamma}X_\alpha,\ \alpha,\beta,\gamma,\delta ... = 1,2,...,n+m, \tag{3.2}$$

$$\mathbf{T}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} + W^\alpha_{\beta\gamma},\ \text{where } [X_\alpha, X_\beta] = W^\gamma_{\alpha\beta}X_\gamma, \tag{3.3}$$

$$\mathbf{R}^\alpha_{\beta\gamma\delta} = X_\delta\Gamma^\alpha_{\beta\gamma} + \Gamma^\varphi_{\beta\gamma}\Gamma^\alpha_{\varphi\delta} - \gamma|\delta + \Gamma^\alpha_{\beta\varphi}W^\varphi_{\gamma\delta}, \tag{3.4}$$

$$\text{Ric}(D) = \mathbf{R}_{\beta\gamma} = \mathbf{R}^\alpha_{\beta\gamma\alpha}, \tag{3.5}$$

$$\mathbf{R} = G^{\alpha\beta}\mathbf{R}_{\alpha\beta}, \tag{3.6}$$

and the eq. (3.1) becomes:

$$\mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{R}G_{\alpha\beta} = \varkappa\mathbf{T}_{\alpha\beta}. \tag{3.7}$$

It results that it is equivalent to the following equations:

$$\begin{cases} R_{ij} - \dfrac{1}{2}(R+S)g_{ij} = \varkappa\mathbf{T}_{ij},\ \overset{1}{P}_{ai} = \varkappa\mathbf{T}_{ai},\ \overset{2}{P}_{ia} = -\varkappa\mathbf{T}ia \\ \\ S_{ab} - \dfrac{1}{2}(R+S)h_{ab} = \varkappa\mathbf{T}_{ab},\ \text{where}: \overset{2}{P}_{ia} = \overset{2}{P}{}_{i\ ka}^{\ k}, \end{cases} \tag{3.8}$$

$$R_{ij} = R^h_{i\ jh},\ \overset{1}{P}_{ai} = \overset{1}{P}{}_{a\ ib}^{\ b},\ S_{ab} = S^c_{abc},\ R = g^{ij}R_{ij},\ S = h^{ab}S_{ab}.$$

All tensor fields from (3.8) are distinguished tensor fields on E i.e. in their laws of transformations to a change of local coordinates, y^α does not appear explicitly.

The conservation law $D_{X_\alpha}(\mathbf{R}^\alpha_\beta - (1/2)\mathbf{R}\delta^\alpha_\beta) = 0$, where $\mathbf{R}^\alpha_\beta = G^{\alpha\gamma}\mathbf{R}_{\gamma\beta}$ can be written as follows:

$$(3.9) \qquad \begin{cases} \left[R^i_j - \frac{1}{2}(R+S)\delta^i_j\right]_{|i} + \overset{1}{P}{}^a_j|_a = 0, \\ \\ \left[S^a_b - \frac{1}{2}(R+S)\delta^a_b\right]|_a - \overset{2}{P}{}^i_{b|i} = 0, \end{cases}$$

where $R^i_j = g^{ik}R_{kj}$,

$$S^a_b = g^{ac}S_{cb},\ \overset{1}{P}{}^a_j = g^{ab}\overset{1}{P}_{bj},\ \overset{2}{P}{}^j_b = g^{ij}\overset{2}{P}_{jb}.$$

Generally, the eqs. (3.9) are not identically satisfied; so it appears that the energy-momentum tensor is not conservative.

Definition 3.1. The eqs. (3.8) will be called the Einstein equations on the total space E of the vector bundle ξ.

4 Some particular cases

a) Let us take $\xi = (TM, \tau, M)$, where M is a generalized Lagrange space i.e. $M = (M^n, g_{ij}(x,y))$ (cf. R. Miron [6]). If some additional conditions on $g_{ij}(x,y)$ are fulfilled (see R. Miron [6]) then $g_{ij}(x,y)$ determines an unique nonlinear connection on (TM, τ, M). Let $(N^i_j(x,y))$ be its local coefficients, $i, j, k, ... = 1, 2, ..., n$, and let (δ_i, ∂_i) be the frame adapted to it. The following Riemannian metric on TM appears as natural:

$$(4.1) \qquad \begin{aligned} &G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j,\ \text{where} \\ &\delta y^i = dy^i + N^i_j dx^j. \end{aligned}$$

As in the general case, a linear $d-$connection on TM is completely determined by a set of functions on TM, let say

$$F\Gamma = (F^i_{jk}, L^i_{jk}, M^i_{jk}, C^i_{jk}).$$

Let J be the natural almost tangent structure on TM i.e.

$$J(\delta_i) = \partial_i, \quad J(\partial_i) = 0.$$

Definition 4.1. A linear $d-$connection D on TM is said to be normal if $DJ = 0$.

It is easy to see that a normal linear $d-$connection is characterized by $L^i_{jk} = F^i_{jk}$ and $M^i_{jk} = C^i_{jk}$, so a normal linear $d-$connection is completely determined by $F\Gamma = (F^i_{jk}, C^i_{jk})$, where F^i_{jk} and C^i_{jk} have the laws of transformation like a linear connection and a tensor on M, respectively, if the local coordinates are changed.

Theorem 4.1. *Given two skew-symmetrical tensor fields T^i_{jk} and S^i_{jk}, there exists an unique metrical normal linear d-connection on TM which has T^i_{jk} and S^i_{jk} as $h(hh)-$ and $v(vv)-$torsions, respectively. Its local coefficients are as follows:*

$$(4.2)\qquad \begin{cases} F^i_{jk} = \dfrac{1}{2}g^{ik}(\delta_j g_{hk} + \delta_k g_{hj} - \delta_h g_{jk} + g_{hr}T^r_{jk} - g_{jr}T^r_{hk} + g_{kr}T^r_{jh}), \\ \\ C^i_{jk} = \dfrac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{hj} - \partial_h g_{jk} + g_{hr}S^r_{jk} - g_{jr}S^r_{hk} + g_{kr}S^r_{jh}). \end{cases}$$

Proof. Taking any linear $d-$connection $F\Gamma = (F^i_{jk}, C^i_{jk})$ and imposing the conditions $g_{ij|k} = 0$, $g_{ij}|_k = 0$, $F^i_{jk} - F^i_{kj} = T^i_{jk}$ and $C^i_{jk} - C^i_{hj} = S^i_{jk}$, one gets that F^i_{jk} and C^i_{jk} are uniquely determined as in (4.2), q.e.d.

Einstein equations associated to the metrical, normal, linear $d-$connection given by the Theorem 4.1 are just the Einstein equations obtained by R. Miron in [6].

b) Preserving the hypothesis from a) we only change the metric G as follows:

$$(4.3)\qquad G = g_{ij}(x,y)dx^i \otimes dx^j - g_{ij}(x,y)\delta y^i \otimes \delta y^j.$$

This G is nondegenerate but it is nondefinite. However, the Theorem 4.1 is still true. The Einstein equations associated to the metrical, normal, linear $d-$connection stated by it, written with respect to the adapted frame (δ_i, ∂_i) are as follows:

$$(4.4)\qquad \begin{cases} R_{ij} - \dfrac{1}{2}(R-S)g_{ij} = \varkappa \mathbf{T}_{ij}, \quad S_{ij} - \dfrac{1}{2}(R-S)g_{ij} = \varkappa \mathbf{T}_{(i)(j)} \\ \\ \overset{1}{P}_{ij} = \varkappa \mathbf{T}_{(i)j}, \quad \overset{2}{P}_{ij} = -\varkappa \mathbf{T}_{i(j)}, \end{cases}$$

where $R_{ij} = R^k_{ijk}$, $R = g^{ij}R_{ij}$.

$S_{ij} = S^k_{ijk}$, $S = g^{ij}S_{ij}$ and in the right hand appear the components of the energy-momentum tensor with respect to the adapted frame.

Remark 4.1. The eqs. (4.4) could be also interesting for physicists because G is nondefinite. Its signature is always (n,n).

Remark 4.2. Setting $P(\delta_i) = -\delta_i$, $P(\partial_i) = \delta_i$ one obtains an almost product structure on TM which satisfies $G(PX, PY) = G(X,Y)$ for any vector fields X and Y on TM and G given by (4.3). Therefore, (TM, P, G) is an almost hyperbolic manifold.

c) Now, let us take $\xi = (E, p, M)$ with dim$\mathbf{F} = 1$. If (U, φ) is a local chart on M, let $(x^1, ..., x^n, x^0)$ the local coordinates of a point $u \in p^{-1}(U)$. These coordinates change as follows (cf. (1.1)):

$$x^{i'} = x^{i'}(x^1, ..., x^n), \; x^{'0} = f(x^1, ..., x^n) \cdot x^0, \tag{4.5}$$

where f is a real function locally defined on M, $f \neq 0$.

The formulas (4.5) show that the manifold E is the most general framework for an unitary projective theory (cf. [8], p. 233).

A nonlinear connection on ξ will be defined by a set of functions (N_i) on E such that $\delta_i = \partial_i - N_i \partial_0$ verify $\delta_{i'} = (\partial'_i, x^i)\delta_i$, where $\partial_0 = \partial/\partial x^0$ and $\delta_{i'} = \delta/\delta x^i$.

A linear $d-$connection will be completely determined by the following set of functions on E, $F\Gamma = (F^i_{jk}, L_k, M^i_j, C)$ where $L_k = L^0_{0k}$, $M^i_j = M^i_{j0}$, $C = C^0_{00}$.

Such a connection has four torsions:

$$T^i_{jk} = F^i_{jk} - F^i_{hj}, \; R_{kh} = \delta_k N_k - \delta_k N_h, \; \overset{1}{P}_j = \partial_0 N_j - L_j, \; \overset{2}{P}{}^i_j = M^i_j \tag{4.6}$$

and four curvatures:

$$\begin{cases} R^i_{jkh} = \delta_h F^i_{jk} + F^l_{ik} F^i_{lh} - h|k + M^i_j R_{kh} \\ \\ \widetilde{R}_{kh} = \delta_h L_k - \delta_k L_h + C R_{kh} \\ \\ \widehat{P}_k = \partial_0 L_k - \partial_k C + \partial_0 (C N_k) \\ \\ \overset{2}{P}{}^i_{jk} = \partial_0 F^i_{jk} - \partial_k M^i_j + \partial_0 (N_k M^i_j) + M^i_h F^h_{jk} - M^h_j F^i_{hk}. \end{cases} \tag{4.7}$$

The $h-$ and $v-$covariant derivatives are defined as in the general case.

Let be $G = g_{ij}(x^1, ..., x^n, x^0) dx^i \otimes dx^i + g_{00}(x^1, ..., x^n, x^0)(\delta x^0)$ where $\delta x^0 = dx^0 + N_i dx^i$, a Riemannian metric on E.

There exists a metrical $d-$connection with T^i_{jk} prescribed. Its local coefficients are as follows:

$$\begin{cases} F^i_{jk} = \dfrac{1}{2} g^{ih} (\delta_j g_{hk} + \delta_k g_{hj} - \delta_k g_{hj} + g_{hr} T^r_{jk} - g_{jr} T^r_{hk} + g_{kr} T^r_{jh}) \\ \\ L_k = \dfrac{1}{2} g^{-1}_{00} \delta_k g_{00}, \quad M^i_j = \dfrac{1}{2} g^{ik} \partial_0 g_{kj}, \quad C = \dfrac{1}{2} g^{-1}_{00} \partial_0 g_{00}. \end{cases} \tag{4.8}$$

Einstein's equations associated to the metrical $d-$connection given by (4.8), written with respect to the adapted frame, are as follows:

$$\begin{cases} R_{ij} - \dfrac{1}{2} R g_{ij} = \varkappa \mathbf{T}_{ij} \\ \\ \overset{1}{P}_i = \varkappa \mathbf{T}_{i0}, \quad \overset{2}{P}{}^k_{jk} = -\varkappa \mathbf{T}_{0j}, \quad R g_{00} = -2 \mathbf{T}_{00}, \end{cases} \tag{4.9}$$

where $R = g^{ij}R_{ij}$.

The conservation law looks as follows:

$$\text{(4.10)} \qquad \begin{cases} R^i_{j|i} - \dfrac{1}{2}R_{|j} + \partial_0(g_{00}^{-1}\overset{1}{P}_j) - g_{00}^{-1}M^i_j\overset{1}{P}_i = 0, \\ \\ \dfrac{1}{2}\partial_0 R + g^{ij}\overset{2}{P}_{j|i} = 0. \end{cases}$$

Remark 4.3. If $(M, g_{ij}(x))$ is a Lorentz manifold and we set $G = g_{ij}(x^1, x^2, x^3, x^4)dx^i \otimes dx^j + g_{00}(x^1, ..., x^4)(\delta x^0)^2$, $i, j = 1, ..., 4$, then the first group from eqs. (4.9) are just the Einstein equations for $(M, g_{ij}(x))$. The following two groups can be thought of or interpreted as Maxwell equations. Therefore, a way for developing an unitary projective theory has appeared.

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Faculty of Mathematics
"Al. I. Cuza" University of Iasi

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MODELS OF FINSLER AND LAGRANGE GEOMETRY

BY

M. ANASTASIEI

1 Introduction

Although closely related to the Riemann geometry, the Finsler geometry had a more slow and sinuous development. Two reasons can he pointed out. Its foundation which is not so firm as of Riemann geometry (a prejudice!) and its too complicated character owing to a lot of differential invariants (a true!). The beginnings were stated by B. Riemann in 1854 (cf. M. Matsumoto [8]). Until 1960 almost all its results had a local character. Since 1960 up to now many efforts to modernize this geometry were made. The theory of connections in fibre bundles has been applied to this aim. In this period the studies in Finsler geometry have progressed very much mainly because three quite distinct models of this geometry were created. These models added to the model "space of line elements" introduced by E. Cartan, enriched considerably the area of researches in Finsler geometry. Our aim is to describe these models. We shall begin by giving a definition of Finsler, as well as of Lagrange geometry. Some historical facts which motivate these definitions are pointed out. The necessity and the usefulness of the models in studying Finsler and Lagrange geometry is explained. The model "space of line elements" will be only sketched since now it is of historical interest. The models which we call "principal Finsler bundle", "vectorial Finsler bundle" and "almost hermitian" will be presented with some details.

It is not our purpose to establish accurately the history of appearance and development of these models. Our lecture is mainly an invitation for studying Finsler and Lagrange geometry by using one of these models. The author is indebted to Prof. Dr. Radu Miron for his helpful advices during the preparation of this lecture.

2 A definition of Finsler and Lagrange geometry

We begin with some historical facts (see M. Matsumoto [83]). In a famous lecture (1854), B. Riemann proposed the study of manifolds endowed with

the so-called Riemannian metric $ds = \sqrt{g_{ij}(x)dx^i dx^j}$. Before arriving at this metric, he is concerned with the concept of generalized metric $ds = L(x^1, ..., x^n,\ dx^1, ...,\ dx^n)$, shortly $ds = L(x, dx)$, which gives the distance between two points x and $x + dx$. B. Riemann himself gives a concrete example of generalized metric $L(x, y) = ((y^1)^4 + ... + (y^n)^4)^{1/4}$ which satisfies the conditions imposed by him:

(L1) $L(x, y) > 0$ for any $y \neq 0$.

(L2) $L(x, ay) = aL(x, y)$ for any $a > 0$.

(L3) $L(x, -y) = L(x, y)$.

Then, the notion of generalized metric space completely had been forgotten for almost 60 years. It was rediscovered in a geometrical treatment of the variational calculus about the beginning of this century. The dissertation of P. Finsler from 1918 is remarkable in this respect. He introduced the so-called fundamental tensor $g_{ij}(x, y) = (\partial^2 L^2/\partial y^i \partial y^j)/2$ and the $C-$tensor $C_{ijk}(x, y) = (\partial g_{ij}(x, y)/\partial y^k)$. The equations $C_{ijk}(x, y) = 0$ characterizes Riemannian metrics among Finslerian metrics. Finsler added a fourth condition on L:

(L4) $g_{ij}(x, y)u^i u^j > 0$ for any $u = (u^i) \neq 0$

This so-called positive definiteness condition is called the regularity condition in the calculus of variations.

A pair (M, L) of an $n-$dimensional manifold M and a general metric L satisfying (L2) and

(L5) $\det(g_{ij}) \neq 0$,

is called a Finsler space. If L satisfies only (L5), the pair (M, L) is called a Lagrange space. This notion was recently introduced by J. Kern [6]. The other conditions (L1, 3, 4) are necessary in some theorems and in some geometrical theories or applications.

The variables y^i, $i = 1, 2, ...n$ from $L(x, y)$ define, from the historical point of view, a direction in the point (x). But these variables can also be thought as parameters and can be taken in a number different by n. This fact is a support for the study of the so-called Finsler geometry of vector bundles (R. Miron [10]).

The entities $g_{ij}(x, y)$ and $C_{ijk}(x, y)$ are not tensors in an usual sense because of their dependence upon y. However, if a change of local coordinates

$$x^{i'} = x^{i'}(x^1, ..., x^n), \tag{2.1}$$

is performed and if suppose that the law of transformation of y is

$$y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i, \tag{2.2}$$

these entitles have laws of transformation similar to that of a tensor of type $(0, 2)$, respectively $(0, 3)$, on manifold M. Such entities are called Finsler tensors. It is noteworthy that if $T(x, y)$ is a Finsler tensor then $\partial T(x, y)/\partial y^i$ is also a Finsler tensor.

The geodesics of a Finsler space (M, L) are the extremals of the variation problem $\delta \int_a^b L(x(t); dx/dt)dt = 0$. These are the solutions of the differential

system

$$\frac{d^2x^i}{dt^2}+\gamma^i_{jk}\left(x,\frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{2.3}$$

where γ^i_{jk} are the Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i.

L. Berwald has put $G^i=\gamma^i_{jk}y^jy^k$ and has considered $G^i_j=\dfrac{\partial G^i}{\partial y^j}$ and $G^i_{jk}=\dfrac{\partial G^i_j}{\partial y^k}$. The laws of transformation of G^i_j and G^i_{jk} are as follows:

$$G^{i'}_{j'}(x',y')=\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}G^i_j+\frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2x^i}{\partial x^{k'}\partial x^{j'}}y^{k'}, \tag{2.4}$$

$$G^{i'}_{j'k'}(x',y')=\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}G^i_{jk}+\frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2x^i}{\partial x^{j'}\partial x^{k'}}. \tag{2.5}$$

So $(G^i_{jk}(x,y))$ changes as a linear connection although it depends by y. The above consideration lead to the following

Definition 2.1. A set of functions of (x,y) whose law of transformation is like that of a geometric object on M is called a Finsler geometric object.

Therefore, $g_{ij}(x,y)$, $C_{ijk}(x,y)$, $G^i_{jk}(x,y)$ are Finsler geometric objects while $G^i_j(x,y)$ is not so. It is now clear what means a field of Finsler geometric objects. These fields can be also defined as cross-sections in convenient fibre bundles (M. Anastasiei [2], R. Miron and M. Anastasiei [15]). But such an abstract definition has a little use without some interpretations of these fields. So appears a necessity to construct the models in which a field of Finsler geometric objects to get a convenient interpretation.

A Finsler vector field is a set of functions $(X^i(x,y))$ with the following law of transformation:

$$X^{i'}(x',y')=\frac{\partial x^{i'}}{\partial x^i}X^i(x,y). \tag{2.6}$$

It is easily to see that $\left(\dfrac{\partial X^i}{\partial x^j}\right)$ does not define a Finsler geometric object. So it is necessary to consider a derivation of X^i which lead to a Finsler object. Such a (covariant) derivative has been defined by L. Berwald:

$$X^i_{;k}=\frac{\partial X^i}{\partial x^k}-G^i_k\frac{\partial X^i}{\partial y^j}+G^i_{jk}X^j \tag{2.7}$$

and is called $h-$covariant derivative. If one puts $\dfrac{\delta}{\delta x^k}=\dfrac{\partial}{\partial x^k}-G^j_k\dfrac{\partial}{\partial y^j}$, then $X^i_{;k}=\dfrac{\delta X^i}{\delta x^k}+G^i_{jk}X^j$, equality which reminds an usual formula for a covariant derivative. So it is natural to define:

$$g_{ij;k}=\frac{\delta g_{ij}}{\delta x^k}-G^s_{ik}g_{sj}-G^s_{jk}g_{is}.$$

If one puts $X^i_{,j} = \dfrac{\partial X^i}{\partial y^j}$ one obtains a new covariant derivative called $v-$covariant derivative. The triad $(G^i_j, G^i_{jk}, 0)$is called the *Berwald connection*. This connection is not metrical because $g_{ij;k} \neq 0$ and $g_{ij,k} \neq 0$, too. To look for a metrical connection one must modify the definition of $h-$ and $v-$covariant derivatives. One defines:

$$X^i_{|k} = \frac{\delta X^i}{\delta x^k} + F^i_{kj} X^j,\ X^i|_k = \frac{\partial X^i}{\partial y^k} + C^i_{kj} X^j, \tag{2.8}$$

where $\dfrac{\delta}{\delta x^k} = \dfrac{\partial}{\partial x^k} - N^j_k \dfrac{\partial}{\partial y^j}$, $N^j_k = F^j_{ik} y^k$, F^i_{jk} is a set of functions of (x, y) which changes like a linear connection on M and C^i_{jk} is a Finsler tensor of type $(1, 2)$ on M.

If the equalities $C^i_{jk} = C^i_{kj}$ and $F^i_{jk} = F^i_{kj}$ are assumed, then from the conditions $g_{ij|k} = 0$ and $g_{ij}|_k = 0$ by a "Christoffel process" one obtains

$$\begin{cases} N^i_j = G^i_j \\ F^i_{jk} = \dfrac{1}{2} g^{is} \left(\dfrac{\delta g_{sk}}{\delta x^j} + \dfrac{\delta g_{sj}}{\delta x^k} - \dfrac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} = \dfrac{1}{2} g^{is} \left(\dfrac{\partial g_{sk}}{\partial y^j} + \dfrac{\partial g_{sj}}{\partial y^k} - \dfrac{\partial g_{jk}}{\partial y^s} \right) = \dfrac{1}{2} g^{is} \dfrac{\partial g_{sj}}{\partial y^k}. \end{cases} \tag{2.9}$$

The triad $(G^i_j, F^i_{jk}, C^i_{jk})$ whose elements are given by (2.9) is called the *Cartan connection* of the Finsler space (M, L). More general, a triad $(N^i_j(x, y)$, $F^i_{jk}(x, y)$, $C^i_{jk}(x, y))$, where N^i_j has a law of transformation similar to C^i_j, F^i_{jk} has a law of transformation similar to G^i_j, F^i_{jk} has a law of transformation similar to G^i_{jk} and C^i_{jk} is a Finsler tensor field, is called a *Finsler connection.* The definition of $v-$ and $h-$covariant derivative is similar to (2.8). The commutation formulae lead to the five torsion Finsler tensors:

$$\begin{aligned} &T^i_{jk} = F^i_{jk} - F^i_{kj},\ R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j},\ P^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - F^i_{kj}, \\ &C^i_{jk},\ S^i_{jk} = C^i_{jk} - C^i_{kj} \end{aligned} \tag{2.10}$$

and three curvature Finsler tensors:

$$\begin{aligned} R^i_{h'jk} &= \frac{\delta F^i_{hj}}{\delta x^k} - \frac{\delta F^i_{hk}}{\delta x^j} + F^r_{hj} F^i_{rk} - F^r_{hk} F^i_{rj} + C^i_{hr} R^r_{jk}, \\ P^i_{hjk} &= \frac{\partial F^i_{hj}}{\partial y^k} - C^i_{hk|j} + C^i_{hr} P^r_{jk}, \\ S_h{}^i{}_{jk} &= \frac{\partial C^i_{hj}}{\partial y^k} - \frac{\partial C^i_{hk}}{\partial y^j} + C^r_{hj} C^i_{rk} - C^r_{hk} C^i_{rj}. \end{aligned} \tag{2.11}$$

Then Bianchi identities can be established and the Finsler spaces with special properties can be studied. We conclude with the following definition of Finsler (Lagrange) geometry.

Definition 2.2. We call Finsler (Lagrange) geometry the study of Finsler geometric objects on a manifold M endowed with a general homogeneous (non-homogeneous) metric L.

The content of the classical Finsler geometry has mainly been obtained by using the methods discussed above. It is full represented by H. Rund's book [19].

3 The model "space of line elements". Nonlinear connections

E. Cartan has arrived at his connection by creating a model of Finsler geometry. As we have seen, the quantities appearing in Finsler geometry depend by $2n$ variables $x = (x^i)$ and $y = (y^i)$. E. Cartan calls the pair (x, y) the supporting element of these quantities and considers the set M' of all the supporting elements. Owing to the homogeneity, y of a supporting element (x, y) is an oriented direction in x so M' is a $(2n-1)-$dimensional manifold called the space of line elements. E. Cartan considers the Finsler geometry as the geometry of the manifold M' and identifies a Finsler connection to an euclidian connection on M'. By using four axioms he determines what is now called the Cartan connection i.e. a metrical Finsler connection completely determined by the fundamental function L (see (2.9)). In this model the Finsler geometric objects are geometric objects defined on M'. Later, instead of M' was considered the total space TM of the tangent bundle over M or $TM - 0$ when the homogeneity is taken in account for.

The models of Finsler geometry created after 1960 have had mainly two purposes. The first one was to give a clear meaning to the notion of Finsler geometric object and the second one was to establish a global definition for connections in Finsler spaces and to re-examine E. Cartan'system of axioms. In all three models which we shall discuss in the following the notion of nonlinear connection appears, in two of them this notion being in a central place. The importance of the nonlinear connections was late recognized. Now there exist a lot of equivalent definitions of this notion. We shall give some here (cf. R. Miron, M. Anastasiei [43]).

1. Let be $\tau : TM \to M$ the tangent bundle to M, τ' its tangent map and $VTM = \ker\tau'$. A nonlinear connection on TM is a subbundle $HTM \subset TTM$ such that $TTM = HTM \oplus VTM$.

2. Let be $\tau^{-1}(TM) \to TM$ the pull-back of TM by τ. Let us denote by $\pi : TTM \to TM$ the projection map. The following sequence of vector bundles over TM is exact:

$$0 \to VTM \xrightarrow{i} TTM \xrightarrow{l=(\pi,\tau')} \tau^{-1}(TM) \to 0.$$

A splitting $\Gamma : \tau^{-1}(TM) \to TTM$ of this exact sequence is a nonlinear connection on TM.

3. Let J be the natural almost tangent structure on TM. A nonlinear connection on TM is a tensor field P of type $(1,1)$ on TM such that $PJ = -J$ and $JP = J$ hold.

4. A nonlinear connection on TM is an almost product structure P on TM which satisfies $P(X) = -X$ for any vertical vector field X.

5. A nonlinear connection is a set of functions (N^i_j) on TM which has the law of transformation like G^i_j in (2.4).

A nonlinear connection always exists if the paracompactness of M is assumed. If (M, L) is a Finsler or a Lagrange space, then there exists a canonical nonlinear connection determined by L only (cf. Section 2).

4 The "principal Finsler bundle" model

As it is well known, a linear connection on a manifold M can be defined as a certain distribution on the linear frame bundle $L(M)$. A point of $L(M)$ is regarded as a pair (x, z) of a point x of M and a frame z in x. The Finsler geometric objects are special functions on TM so they depend on (x, y), a pair of a point x of M and a tangent vector y at x. Therefore, the set of triads (x, y, z) may be a good foundation for Finsler geometry. Such a set is obtained as follows. Let $\tau^{-1}(L(M)) \to TM$ be the pull-back of $L(M)$ by τ. This is a principal bundle over TM with structural group $GL(n, R)$. It is called the Finsler bundle of M and it will be denoted by $F(M)$. Its total space is $F = \{(y, z) \in TM \times L(M), \tau(y) = \pi(z)\}$. A right translation β_g of $F, g \in GL(n, R)$ is given as: $u = (y, z) \to ug = (y, zg)$. The bundle $F(M)$ was introduced by L. Auslander ([4]). It was also considered by H. Akbar-Zadeh [1]. $F(M)$ was called a Finsler bundle of M by M. Matsumoto. He also used it systematically and efficiently as a model of Finsler geometry (see his monograph [7]). Let $(R^n)^r_s$ be the space of tensors of type (r, s) over R^n. A Finsler tensor field of type (r, s) is defined as a map $K : F \to (R^n)^r_s$ which satisfies a condition $K \circ \beta_g = g^{-1}K$ for any $g \in GL(n, R)$. This definition is equivalent to a classical one (M. Matsumoto [7], p. 49). A Finsler connection F on a manifold M is a pair (Γ, N) of a connection Γ in $F(M)$ and a nonlinear connection M on the tangent bundle TM. Such a definition is very general because Γ and N are not yet related. A second definition of a Finsler connection ([7], p. 63) lead to a definition of $v-$ and $h-$covariant derivatives quite similar to that with respect to a linear connection. Torsions and curvatures are obtained by a suitable generalization of the structure equations of a linear connection to a Finsler connection ([7], p. 70-76).

All Bianchi identities are obtained from some general identities. If M is endowed with a fundamental function L, then the nonlinear connection is completely determined by L. The following theorem of M. Matsumoto holds:

The Cartan connection $C\Gamma$ of a Finsler space (M, L) is uniquely determined by the five axioms as follows:

(1) $h-$metrical: $g_{ij\ |k} = 0$

(2) without $h-$torsion: $T = 0$ $(F^i_{jk} = F^i_{kj})$

(3) $v-$metrical: $g_{ij}|_k = 0$

(4) without $v-$torsion: $S^1 = 0$ $(C^i_{jk} = C^i_{kj})$

(5) $D^i{}_j = N^i_j - F^i_{kj}y^k = 0$ (the deflection tensor vanishes).

The "principal Finsler bundle" model lead to a clear definition of the notion of Finsler geometric object, to a global definition of Finsler connections

from which all classical Finsler connections are derived and to an interesting theory of transformations of Finsler spaces. Of course, there are some problems which can not be solved nor attacked by using this model. For instance the theory of Finsler spaces with constant curvature or the theory of subspaces in Finsler spaces.

5 The "vectorial Finsler bundle" model

In this model the base manifold is TM, too. The pull-back $\tau^{-1}(TM) \to TM$ of TM by τ will be called the vectorial Finsler bundle of M. If (x^i) is a coordinate system on M and (x^i, y^i) is the coordinate system on TM induced by it, then $\left(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^i}\right)$ is a basis in $T_u TM$, $u \in TM$, and $\left(\dfrac{\partial}{\partial y^i}\right)$ is a basis in $\tau^{-1}(TM)$, the fiber of $\tau^{-1}(TM)$ over u. It follows easily that the vectorial Finsler bundle is isomorphic to the vertical subbundle. We denote by v its inclusion map in TTM. A cross-section $\overline{X}$ of the vectorial Finsler bundle has the local form $\overline{X} = \overline{X}^i \dfrac{\partial}{\partial y^i}$. Since $\dfrac{\partial}{\partial y^i} = \dfrac{\partial x^{i'}}{\partial x^i} \dfrac{\partial}{\partial y^{i'}}$ it follows that the set of functions $(\overline{X}^i(x, y))$ defines a Finsler vector field. More general, the tensorial algebra on the vectorial Finsler bundle is a model for the algebra of Finsler tensor fields.

A Finsler connection is defined as a regular connection in the vectorial Finsler bundle. Let be $\nabla : \mathcal{X}(TM) \times S(\tau^{-1}(TM)) \to S(\tau^{-1}(TM))$, $(X, Y) \to \nabla_X Y$ a linear connection in the vectorial Finsler bundle. Let be $C = y^i \dfrac{\partial}{\partial y^i}$ the canonical field (Liouville) on TM. A vector field X on TM is called horizontal if $\nabla_X C = 0$. Let be H_u the subspace of horizontal vectors and V_u the subspace of vertical vectors. The connection ∇ is called regular if $T_u TM = H_u \oplus V_u$ for any $u \in TM$. Such a decomposition of $T_u TM$ defines a splitting of the exact sequence from Section 3, hence a nonlinear connection on TM. If ∇ is a regular connection, then τ' is an isomorphism on H_u. Let us denote by h_u the map $(\tau'/H_u)^{-1} : T_{\tau(u)}M \to H_u$ and let us put $h_u\left(\dfrac{\partial}{\partial x^i}\right) = \dfrac{\delta}{\delta x^i}$. It results $\tau'_u\left(\dfrac{\delta}{\delta x^i} - \dfrac{\partial}{\partial x^i}\right) = 0$ because of $\tau'_u \circ h_u =$identity and of $\tau'\left(\dfrac{\partial}{\partial x^i}\right) = \dfrac{\partial}{\partial x^i}$. Therefore $\dfrac{\delta}{\delta x^i} - \dfrac{\partial}{\partial x^i}$ are vertical vector fields. We may write $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N_i^j \dfrac{\partial}{\partial y^j}$ because V_u is spanned by $\left(\dfrac{\partial}{\partial y^j}\right)$. A linear connection on the vectorial Finsler bundle is locally given as follows:

$$\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial y^j} = \Gamma^i_{jk} \frac{\partial}{\partial y^i}, \ \nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = C^i_{jk} \frac{\partial}{\partial y^i}.$$

From the equality $\nabla_{\frac{\partial}{\partial y^i}} C = (\delta^j_i + y^k C^j_{ki})$ it follows that ∇ is regular if and only if the matrix $(\delta^j_i + y^k C^j_{ki})$ is regular for any y. The condition $\nabla_{\frac{\delta}{\delta x^i}} C = 0$ is equivalent to $N^k_i(\delta^i_k + y^s C^j_{sk}) = \Gamma^j_{si} y^s$. It follows again that the regularity

condition on ∇ allow the determination of a nonlinear connection (N^i_j). If we put $F^i_{jk} = \Gamma^i_{jk} - N^p_k C^i_{jp}$ then it results that $(N^i_j, F^i_{jk}, C^i_{jk})$ defines a Finsler connection in the classical sense. Therefore, any regular connection in the vectorial Finsler bundle is a model of a Finsler connection. The curvatures and torsions of such a Finsler connection are obtained as follows.

Let be $\widetilde{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ the curvature of ∇. The following tensor fields: $R(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(h\bar{X},h\bar{Y})\bar{Z}$, $P(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(v\bar{X},h\bar{Y})\bar{Z}$, $S(\bar{X},\bar{Y})\bar{Z} = \widetilde{R}(v\bar{X},v\bar{Y})\bar{Z}$ are Finsler tensor fields and are models for the curvatures of a Finsler connection. The tensor field $T(X,Y) = \nabla_X \ell Y - \nabla_Y \ell X - \ell[X,Y]$ is called the torsion of ∇. It results that $T(hX,hY)$, $T(hX,vY)$ and $T(vX,vY)$ are Finsler tensor fields, models for the three torsions of a Finsler connection. The others are C^i_{jk} and R^i_{jk} given by $\left[\dfrac{\delta}{\delta x^j}, \dfrac{\delta}{\delta x^k}\right] = R^i_{kj}\dfrac{\partial}{\partial y^i}$.

The model of a metrical Finsler connection is a regular connection ∇ which verifies $\nabla g = 0$. We remark that as a model of Finsler geometry can also serve the vertical subbundle which is isomorphic to the vectorial Finsler bundle. This model appears in a paper by V. Oproiu [17]. The "vectorial Finsler bundle" model was systematically used by B.T. Hassan [5]. A generalization of it was treated by D. Opris [16].

6 The "almost hermitian" model

This model was pointed out by R. Miron and was used by him for an interesting theory of finslerian relativity (R. Miron [12]). The base manifold is TM furnished with a nonlinear connection determined by the fundamental function or not. The tensorial Finsler fields have as models the elements of the tensorial algebra of the bundle $H \oplus V \to TM$. Let $\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$ be the local frame adapted to the decomposition $T_uTM = H_u \oplus V_u$. Putting $F\left(\dfrac{\delta}{\delta x^i}\right) = -\dfrac{\partial}{\partial y^i}$, $F\left(\dfrac{\partial}{\partial y^i}\right) = \dfrac{\delta}{\delta x^i}$, one obtains an almost complex structure $(F^2 = -I)$ on TM. A model for a Finsler connection is a linear connection D on TM which satisfies the following two conditions:

(1) D preserves by parallelism the horizontal distribution $u \to H_u$ as well as the vertical distribution $u \to V_u$.

(2) $DP = 0$.

Indeed, the first condition leads to the following local form of D:

$$D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = F^i_{jk}\frac{\delta}{\delta x^j},\ D_{\frac{\partial}{\partial y^k}}\frac{\delta}{\delta x^j} = \widetilde{C}^i_{jk}\frac{\delta}{\delta x^i},$$

$$D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^j} = \widetilde{F}^i_{jk}\frac{\partial}{\partial y^i},\ D_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j} = C^i_{jk}\frac{\partial}{\partial y^i},$$

and the second one gives $\widetilde{F}^i_{jk} = F^i_{jk}$, $\widetilde{C}^i_{jk} = C^i_{jk}$. So D which satisfies (1) and (2) is a model for the Finsler connection $(N^i_j, F^i_{jk}, C^i_{jk})$. As a model of the

fundamental tensor (g_{ij}) is taken the tensor field $G = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$, where $\delta y^i = dy^i + N^i_k dx^k$. This G is called the $N-$lift of (g_{ij}). By a direct calculation it follows that $G(FX, FY) = G(X, X)$ for any vector fields X, Y on TM. Therefore (F, G) defines an almost hermitian structure on TM. The triad $H^{2n} = (TM, G, F)$ is called the "almost hermitian" model of a Finsler space (M, L) or a Lagrange space $(M, \mathcal{L})$. The term is also justified by the following theorem:

Let G be a Riemannian metric on TM of rank n on the vertical distribution and let N be the distribution supplementary and orthogonal to the vertical distribution. Let F be the almost complex structure determined by N. If (G, F) is an almost hermitian structure then there exists an unique fundamental tensor (g_{ij}) whose $N-$lift is G.

Proof. The distribution N spanned by $\dfrac{\delta}{\delta x^i}$ is determined from the equations $G\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^j}\right) = 0$. Locally, G is as follows: $G = h_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$. From $G(FX, FY) = G(X, Y)$ it results $h_{ij} = g_{ij}$ is obvious that the $N-$lift of g_{ij} is just G.

The following theorem holds (R. Miron [12]):

The model H^{2n} is hermitian if and only if

$$R^i_{jk} = \frac{\delta N^i_k}{\delta x^j} - \frac{\delta N^i_j}{\delta x^k} = 0, \; t^i_{jk} = \frac{\partial N^i_k}{\partial y^j} - \frac{\partial N^i_j}{\partial y^k} = 0.$$

A metrical Finsler connection is a linear connection D on TM which satisfies and the third condition:

(3) $DG = 0$.

There exists an unique metrical Finsler connection such that $T(hX, hY) = 0$ and $T(vX, yY) = 0$, where T is its torsion.

This connection coincide to the Cartan connection when the nonlinear connection N is determined by the fundamental function L. On TM there exists also a symplectic structure defined by $\phi(X, Y) = G(X, FY)$. It is easy to see that a metrical Finsler connection is also a symplectic one ($D\phi = 0$). If M is a Finsler space its model H^{2n} is almost Kähler i.e. $d\phi = 0$ (Matsumoto [9]). This result is also valid if M is a Lagrange space (V. Oproiu [18]).

The "almost hermitian" model suggests at least two generalizations studied until now. The first one is the considering of the linear connections D on TM which preserve the horizontal and vertical distributions but do not verify $DF = 0$. Locally, such a connection has four distinct components. A Lagrangian theory of relativity by using such a connection was developed (R. Miron, S. Watanabe, S. Ikeda [13]). The second one is the considering of the geometry of the total space of a vector bundle (R. Miron [10]) or a principal bundle (M. Anastasiei [3]).

Quite recently it was observed the importance of the study of the pair $(M, g_{ij}(x, y))$, where $g_{ij}(x, y)$ is not provided by a fundamental function L (R. Miron [10]). The models described above can also be used for studying such a spaces $(M, g_{ij}(x, y))$ called generalized Lagrange spaces (R. Miron [12]).

The "almost hermitian" model is very complex so it allow to obtain much information about Finsler and Lagrange spaces.

The models which we just described rise a lot of new problems for Finsler and Lagrange geometry and allow the solving of the elder problems which could not be solved in a classical treatment. These models suggests also various generalizations and applications of Finsler and Lagrange geometry.

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Faculty of Mathematics
"Al. I. Cuza" University of Iasi

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CONSERVATION LAWS IN THE $\{V, H\}-$BUNDLE MODEL OF RELATIVITY

Dedicated to the Memory of
Professor Dr. Akitsugu Kawaguchi, Founder of the Tensor Society

by Mihai ANASTASIEI

1 Introduction

In a recent paper [1][1] we have considered the Einstein equations on the total space of a vector bundle in order to obtain a Finslerian unitary projective theory as an extension of the Finslerian theory of relativity developed by R. Miron [5]. We have written the corresponding conservation laws which here, generally, are not identities, i. e., it is of interest to find vector bundles for which such a conservation laws are identically verified. In this paper we take, into considerations, the vector bundles whose type fibers are finite dimensional normed spaces V and, moreover, admit the reductions to a subgroup H of the group of the automorphisms of V which preserve the norm, shortly, $\{V, H\}-$bundles. This class of vector bundles, which contains the tangent bundles to the $\{V, H\}-$manifolds of Y. Ichijyo [2], may be of own interest so we treat it with some details in §3. In §2 we give necessary preliminaries from the geometry of the total space of a vector bundle (cf. [7], [8]). The conservation laws on the $\{V, H\}-$bundles are discussed in §4.

2 Vector bundles. Metrical $d-$connections

Let M be an $n-$dimensional differentiable (of class C^∞) manifold and $\xi = (E, p, M)$, $p : E \to M$, a vector bundle whose type fiber is a vector space V isomorphic to R^m. Let $\{(U_a, \widetilde{\phi}_a, R^n)\}$ be an atlas on M and let $\{(U_a, \phi_a, V)\}$ be a bundle atlas of ξ i.e. $\phi_a : p^{-l}(U_a) \to U_a \times V$ are bijective mappings such that $p\phi_a^{-1}(x, v) = x$ for $x \in M$ and $v \in V$, and the applications $g_{\beta,x}(x) = \phi_{\beta\alpha} \circ \phi_{\alpha,x}^{-1} : U_\alpha \cap U_\beta \to V$ are differentiable. The manifold structure of E is defined by the differentiable atlas $\{(p^{-1}(U_\alpha), h_\alpha)\}$, where $h_\alpha : p^{-l}(U_\alpha) \to R^n \times V$ is given as $h_\alpha(u) = (\widetilde{\phi}_\alpha(u), \phi_{\alpha,p(u)}(u))$. If we set $h_\alpha(u) = (x^h, y^a)$ and

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[1]Number in brackets refer to the references at the end of the paper

$h_\beta(u) = (x^{k'}, y^{a'})$, $k, k' = 1, ..., n$; $a, a' = 1, ..., m$, then $h_\beta \circ h_\alpha^{-1}$ is as follows:

$$
(2.1) \qquad \begin{cases} x^{k'} = x^{k'}(x^1, ..., x^n), \det\left(\dfrac{\partial x^{k'}}{\partial x^k}\right) \neq 0, \\ \\ y^{a'} = y^a \cdot S_a^{a'}(x^1, ..., x^n), \ \|S_a^{a'}(x)\| \in GL(m, R). \end{cases}
$$

The transformation law (2.1) of the local coordinates on E shows that E, for $m = 1$ or $m = 2$, is the most general framework for an unitary projective theory (cf. [10] p. 233).

Let $p^T : TE \to TM$ be the differential of p. Then $\ker p^T = VE$ is a subbundle of $TE \to TM$ called the *vertical subbundle*.

Definition 2.1. A nonlinear connection on ξ is a subbundle HE of $TE \to TM$ such that $TE = HE \oplus VE$.

The local fiber H_uE of the vector bundle $HE \to TM$ is spanned by (δ_k) given as follows:

$$
(2.2) \qquad \delta_k = \partial_k - N_k^a(x, y)\partial_a.
$$

Here ∂_k and ∂_a stand for $\dfrac{\partial}{\partial x^k}$ and $\dfrac{\partial}{\partial y^a}$, respectively. The functions $N_k^a(x, y)$ are called the *local coefficients of the nonlinear connection*. This set of functions has appeared, for the particular case $E = TM$, early in the development of Finsler geometry, but the first who recognized its importance and treated it as defining a nonlinear connection was A. Kawaguchi ([3], [4]). The mapping $u \to H_uE$ ($u \to V_uE$) is called the horizontal (vertical) distribution. The local frame (δ_k, ∂_a) is adapted to the horizontal and the vertical distributions. Its dual is $(dx^k, \delta y^a)$, where $\delta y^a = dy^a + N_k^a dx^h$. The tensorial algebra spanned by $1, \delta_k, \partial_a, dx^k, \delta y^a$ is called the *algebra of d-tensor fields* ([6], [7]).

Definition 2.2. A linear connection D on E is a $d-$connection if it preserves over parallel displacement of the horizontal and the vertical distributions.

Every $d-$connection D on E can be locally given as follows:

$$
(2.3) \qquad \begin{cases} D_{\delta_k}\delta_j = F_{jk}^i(x, y)\delta_i, \quad D_{\delta k}\partial_b = L_{bk}^a(x, y)\partial_a, \\ \\ D_{\partial_a}\delta_j = M_{ja}^i(x, y)\delta_i, \quad D_{\partial_c}\partial_b = C_{bc}^a(x, y)\partial_a. \end{cases}
$$

The coefficients $F_{jk}^i(x, y)$ and $L_{bk}^a(x, y)$ change under (2.1) like the coefficients of a linear connection on M and on ξ, respectively, although they depend on y^a; $M_{ja}^i(x, y)$ and $C_{bc}^a(x, y)$ are defining tensor fields on E. Conversely, a set of coefficients $L\Gamma = (F_{jk}^i(x, y),\ L_{bk}^a(x, y),\ M_{ja}^i(x, y),\ C_{bc}^a(x, y))$ which change under (2.1) as the above determines an unique $d-$connection on E. We shall denote by $_|$ and $|$ the $h-$ and $v-$covariant derivative associated with the $d-$connection D (see [6]). The commutation or Ricci formulae introduce for a $d-$connection five torsion $d-$tensor fields:

$$
(2.4) \qquad \begin{cases} T^i{}_{jk} = F_{jk}^i - F_{kj}^i, \quad R^a{}_{jk} = \delta_k N_j^a - \delta_j N_k^a, \\ \\ \overset{1}{P}{}^a{}_{jb} = \partial_b N_j^a - L_{bj}^a, \quad \overset{2}{P}{}^i{}_{jb} = M_{jb}^i, \ S^a{}_{bc} = C_{bc}^a - C_{cb}^a, \end{cases}
$$

and six curvature $d-$tensor fields:

$$
(2.5) \qquad \begin{cases}
R_{j\ kh}^{\ i} = \delta_h F^i_{jk} + F^l_{jk} F^i_{lh} - k/h + M^i_{ja} R^a_{\ kh}, \\
\widetilde{R}_{b\ kh}^{\ a} = \delta_h L^a_{bk} + L^c_{bk} L^a_{ch} - k/h + C^a_{bc} R^c_{kh}, \\
\overset{1}{P}{}_{b\ kc}^{\ a} = \partial_c L^a_{bk} - C^a_{bc|k} + C^a_{bd} P^d_{\ kc}, \\
\overset{2}{P}{}_{j\ kc}^{\ i} = \partial_c F^i_{jk} - M^i_{jc|k} + M^i_{jb} P^d_{\ kc}, \\
M^i_{j\ bc} = \partial_c M^i_{jb} + M^h_{jb} M^i_{hc} - b/c, \\
S_{b\ cd}^{\ a} = \partial_d C^a_{bc} + C^e_{bc} C^a_{ed} - c/d,
\end{cases}
$$

where $k/h, b/c, c/d$ mean the interchange of indices in the foregoing terms.

A metrical structure on E is a tensor field G of type $(0,2)$ on E, symmetric and nondegenerate. It determines an unique nonlinear connection on ξ by taking into consideration the distribution which is orthogonal to the vertical distribution, with respect to it. In the frames adapted to this nonlinear connection, G can be expressed as follows:

$$
(2.6) \qquad G(x,y) = g_{ij}(x,y) dx^i dx^j + \widetilde{g}_{ab}(x,y) \delta y^a \delta y^b.
$$

A $d-$connection D on E is metrical with respect to G if $DG = 0$. It is easy to prove that a $d-$connection D is metrical if and only if

$$
(2.7) \qquad g_{ij|k} = 0, \quad g_{ij}|_a = 0, \quad \widetilde{g}_{ab}|_c = 0, \quad \widetilde{g}_{ab|k} = 0
$$

hold. There exists a metrical $d-$connection which has the torsion fields T^i_{jk} and S^a_{bc} prescribed and which is unique in a certain sense [1]. Another metrical $d-$connection will be constructed in §4.

3 $\{V, H\}-$bundles

Let ξ be the vector bundle from §2. Suppose that its fiber V is endowed with a norm $\|\cdot\| : V \to R_+$, i.e. V is a Minkowski space. If $v = v^a e_a$, where (e_a) is a basis of V, we set $\|v\| = f(v^l, ..., v^m) = f(v^a)$ and suppose that f is differentiable at least of class C^3 for $v \neq 0$. The set $\{T | T \in GL(m,R),\ \|Tv\| = \|v\|, v \in V\}$ is a Lie group. Let H be a subgroup of it.

Definition 3.1. A vector bundle $\xi = (E, p, M)$ is said to be a $\{V, H\}-$bundle if there exists a bundle atlas $\{(p^{-l}(U_\alpha), \phi_\alpha, V)\}$ such that the mappings $\psi_{\beta,x} \circ \psi^{-1}_{\alpha,x}$ belongs to H for every $x \in U_\alpha \cap U_\beta \neq \psi$. We also say that ξ admits an $H-$structure.

Proposition 3.1. *If ξ is a $\{V, H\}-$bundle, then its local fibers are Minkowski spaces isomorphic and isometric each to others.*

Proof. If $u \in E_x$, we set $\|u\| = f(\psi_{\alpha,x}(u))$ and obtain a norm on E_x which does not depend on $\psi_{\alpha,x}$ because ξ admits an $H-$structure. Namely $\psi_{\alpha,x}$ is also an isometry of E_x and V for every $x \in M$. Therefore the local fibers are isomorphic and isometric each to others. Q. E. D.

Examples a) If V is a Euclidian space then $O(m)$ leaves invariant its norm. Then ξ is a $\{V, O(m)\}-$bundle if and only if it is a Riemannian bundle.

b) If $\xi = (E = TM, \tau, M)$ and M is modeled by V, then ξ is a $\{V, H\}-$bundle if and only if M is a $\{V, H\}-$manifold in Ichijyo's sense [2].

If $\{(p^{-1}(U_\alpha), \phi_\alpha, V)\}$ is any bundle atlas on ξ, the cross-section $s_{\alpha,a}(x) = \phi^{-1}(x, e_a)$ define a frame in E_x and the fiber coordinates (y^a) are introduced by the equality $u_x = y^a s_{\alpha,a}(x)$. Setting $\sigma_{\alpha,a}(x) = \psi_\alpha^{-1}(x, e_a)$ we obtain a new frame in E_x, so that $u_x = u^a \sigma_{\alpha,a}(x)$. Taking $\sigma_{\alpha,a}(x) = \lambda_a^b(x) s_{\alpha,b}(x)$, it follows $y^a = \lambda_b^a(x) u^b$ or $u^a = \mu_b^a(x) y^b$, where (μ_b^a) is the inverse of the matrix (λ_b^a). Now, $\|u_x\| = f(\psi_{\alpha,x}(u)) = f(\psi_{a,x}(u^a \sigma_{\alpha,a}(x)) = f(u^a) = f(\mu_b^a(x) y^b)$. Now we have a function $F : E \to R_+$, given locally by $F(x, y) = f(\mu_b^a(x) y^b)$, which is $(1)-$homogeneous and differentiable at least of class C^3 for $y \neq 0$. Moreover, as F is provided by a norm, the matrix $(h_{ab}(x, y)) = \left(\frac{1}{2}\partial_a \partial_b (F^2)\right)$ is nonsingular, and the quadratic form $h_{ab}\eta^a \eta^b$ is positive defined (see [9] p. 21 for a proof). We say that F is a *fundamental Finsler function* on E. Therefore we have proved

Theorem 3.1. *If a vector bundle (E, p, M) admits an $H-$structure, then there exists on E a fundamental Finsler function of the form $F(x, y) = f(\mu_b^a(x) y^b)$.*

Definition 3.2. A linear connection ∇ on a $\{V, H\}-$bundle is said to be an *$H-$connection* if its parallel displacement preserves the Minkowski norms of fibers.

Let us set $\nabla_{\partial_k} s_a = \Gamma_{ak}^b(x) s_b$. We have

Theorem 3.2. *If ∇ is an $H-$connection on the $\{V, H\}-$bundle ξ, then $\overset{\circ}{\delta}_k F = 0$, where $\overset{\circ}{\delta}_k = \partial_k - \Gamma_{bk}^a(x) y^b \partial_a$.*

Proof. Let $C = \{x(t),\ t \in [0, 1]\}$ be a curve on M and $S(x(t)) = S^a(x(t)) s_a(x(t))$ a cross-section of ξ along C. It is parallel along C with respect to ∇ if and only if $\nabla_{\dot{x}(t)})S = 0$, i.e. $\dfrac{dS^a}{dt} + \Gamma_{bk}^a(x) S^b \dfrac{dx^k}{dt} = 0$. If ∇ is an $H-$connection, then $\dfrac{d\|S(t)\|}{dt} = 0$. But $\|S(t)\| = F(x(t), S^a(x(t)))$ so we obtain $0 = \partial_k F \dfrac{dx^k}{dt} + \partial_a F \dfrac{dS^a}{dt} = (\partial_k F - \Gamma_{bk}^a(x) S^b) \dfrac{dx^k}{dt} = \overset{\circ}{\delta}_k F \dfrac{dx^k}{dt}$. Since C is arbitrary, it results $\overset{\circ}{\delta}_k F = 0$. Q. E. D.

4 Einstein equations and the conservation laws

If $\Gamma_{bk}^a(x)$ are the coefficients of a linear connection ∇ on ξ, then $\overset{\circ}{N}{}_k^a(x, y) = \Gamma_{bk}^a(x) y^b$ define a nonlinear connection on ξ. We consider a "deformation"

of this nonlinear connection, i.e. $N^a_k(x,y) = \overset{\circ}{N}{}^a_k(x,y) + A^a_k(x)$, where $A^a_k(x)$ defines a $d-$tensor field on E, depending only on x. Using Theorem 3.2 we easily obtain

Proposition 4.1. *If ∇ given by $\Gamma^a_{bk}(x)$ is an $H-$connection on the $\{V, H\}-$bundle ξ then, $\delta_k F = \partial_k F - N^a_k \partial_a F = 0$ if and only if*

$$A^a_k(x)\partial_a F = 0, \tag{4.1}$$

holds good.

In what follows we assume that ∇ is an $H-$connection on ξ and that $A^a_k(x)$ satisfies (4.1). Now, let $g_{ij}(x)$ be a metric on M and let $\Gamma^i_{jk}(x)$ be the corresponding Christoffel symbols. The definition of F shows that in the adapted frames to the $H-$structure on ξ, the functions h_{ab} depend only on y. The following natural metrical structure on E can be considered:

$$\mathcal{G}(x,y) = g_{ij}(x)dx^i dx^j + h_{ab}(y)\delta y^a \delta y^b. \tag{4.2}$$

This is an example of Riemann-Minkowski metric on E. A study of the Riemann-Minkowski metrics on TM is given in [8]. Let $C^a_{bc}(y)$ be the Christoffel's symbols associated with $h_{ab}(y)$. Then it is clear that $E\Gamma = (\Gamma^i_{jk}, \Gamma^a_{bk}(x), 0, C^a_{bc}(y))$ is a $d-$connection on E. Moreover, we have

Theorem 4.1. *The $d-$connection $E\Gamma$ is metrical with respect to the metric $\mathcal{G}$.*

Proof. The first three equalities from (2.7) hold by virtue of the definition of $E\Gamma$. To prove the last one we remark that $\delta_k F = 0$ is the same with $F_{|k} = 0$ and we note that $|k \circ \partial_a = \partial_a \circ |k$. So, $h_{ab|k} = \partial_a \partial_b (F_{|k}) = 0$. Q.E.D.

An easy computation shows that

$$R^a{}_{jk} = R_b{}^a{}_{jk}(x)y^b + (\partial_k A^a_j + \Gamma^a_{bk}A^b_j - j/k), \tag{4.3}$$

where $R_b{}^a{}_{jk}(x)$ is the curvature of ∇. The others torsions of $E\Gamma$ are vanishing. The $d-$tensor field $A^a_k(x)$ can be viewed as defining an 1-form A on M, valued in ξ. If $\overset{\circ}{\nabla}$ denotes the Levi-Civita connection of g_{ij} then the covariant differential of A is the 2-form

$$(\widetilde{\nabla}A)(X,Y) = \nabla_Y A(X) - A(\overset{\circ}{\nabla}_Y X), \text{ for } X, Y \in \mathcal{X}(M).$$

Theorem 4.2. *The metrical $d-$connection $E\Gamma$ coincide with the Levi-Civita connection of $\mathcal{G}$ if and only if a) ∇ is flat, and b) $\widetilde{\nabla}A$ is symmetric.*

Proof. By annihilating $R^a{}_{jk}$ from (4.3) we obtain that ∇ is flat, and $\widetilde{\nabla}A$ is symmetric. The converse is clear. Q. E. D.

Particularizing (2.5) one obtains

Proposition 4.2. *The curvatures of $E\Gamma$ are as follows: $R_j{}^i{}_{kh}$ is the curvature of $\overset{\circ}{\nabla}$, $\widetilde{R}_b{}^a{}_{kh} = R_b{}^a{}_{kh} + C^a_{bc}R^c{}_{kh}$, $\overset{1}{P}{}_b{}^a{}_{kc} = -C^a_{bc|k} = 0$, $\overset{2}{P}{}_j{}^i{}_{kc} = 0$, $S_b{}^a{}_{cd}$ has the general form.*

Note that $C^a_{bc|k} = 0$ results from $C^a_{bc} = h^{ad}\partial_b\partial_c F^2/4$ and $F_{|k} = 0$.

Corollary 4.1. *The metrical $d-$connection $E\Gamma$ has no curvature if a) $\overset{\circ}{\nabla}$ is flat, b) ∇ is flat, c) $\widetilde{\nabla}A$ is symmetric, d) $S_b{}^a{}_{cd} = 0$.*

As to the metrical $d-$connection E we are associated with the Einstein equation

$$\mathrm{Ric}(E\Gamma) - \mathcal{R}\mathcal{G} = \varkappa\mathcal{T}, \tag{4.4}$$

where $\mathrm{Ric}(E\Gamma)$ and $\mathcal{R}$ denote the Ricci tensor and the scalar curvature of $E\Gamma$, respectively; $\varkappa$ is a constant; $\mathcal{T}$ is the energy momentum tensor. With respect to the adapted frame (δ_k, ∂_a), equation (4.4) decomposes as follows (cf. [1]):

$$\begin{cases} R_{ij} - \dfrac{1}{2}(R+S)g_{ij} = \varkappa\mathcal{T}_{ij},\ 0 = \mathcal{T}_{ai},\ 0 = \mathcal{T}_{ia} \\ \\ S_{ab} - \dfrac{1}{2}(S+R)h_{ab} = \varkappa\mathcal{T}_{ab}, \end{cases} \tag{4.5}$$

where $R_{ij} = R_i{}^k{}_{jk}$, $S_{ab} = S_a{}^c{}_{bc}$, $R = g^{ij}R_{ij}$, $S = h^{ab}S_{ab}$; in the right members appear the components of $\mathcal{T}$, two of them must be taken zero because the curvatures $\overset{1}{P}$ and $\overset{2}{P}$ of $E\Gamma$ are vanishing.

Equation (4.5) will be called the *Einstein equations* on E. The conservation law is obtained by annihilating the divergence of the tensor which appear as the first member of (4.4), called the Einstein tensor. In the adapted frame one obtains as conservation laws:

$$\begin{cases} \left(R^i_j - \dfrac{1}{2}(R+S)\delta^i_j\right)_{|i} = 0, \\ \\ (S^a_b - (R+S)\delta^a_b)|_a = 0. \end{cases} \tag{4.6}$$

As is well known the divergence of the Einstein tensor associated with the Levi-Civita connection identically vanishes. By using Theorem 4.2 one obtains

Theorem 4.3. *The conservation laws on E with respect to $E\Gamma$ are identities if: i) ∇ is flat and ii) $\widetilde{\nabla}A$ is symmetric.*

The conditions ii) and (4.1) on A can be easily satisfied taking, for instance, $A = 0$. The condition i) is a strong one because if M is simply connected, then $E = M \times V$. Examining (4.6) we shall find algebraic conditions on A under which the conservation laws are identities. The second equality (4.6) reduces to $\left(S^a_b\dfrac{1}{2}S\delta^a_b\right)|_a = 0$ which is an identity by virtue of Bianchi identities. The first is reduced to $\left(R^i_j\dfrac{R^{\,i}}{\delta\;j}\right)S_{|i} = 0$, and by virtue of the Bianchi identities it become an identity if and only if $S_{|i} = 0$. Namely we have proved

Theorem 4.4. *The conservation laws on E with respect to $E\Gamma$ are identities if*

$$(\Gamma^a_{bk}y^b + A^a_k)\partial_a S = 0, \tag{4.7}$$

holds good.

The conditions (4.1) and (4.7) form together an algebraic system of $2n$ equations with nm unknowns A^a_k. If $m = 1$ the first n equations give $A^1_k = 0$, $k = 1, ..., n$ and the last n equations are verified if S is constant or if $\Gamma^{\,1}_{1k} = 0$. For $m = 2$ the determinant of the system is $-(\partial_1 F \partial_2 S - \partial_2 F \partial_1 S)^n$ which generally is different from zero. For $m > 2$ some unknowns can be arbitrarily taken.

To conclude we must say that the total space E of a $\{V, H\}-$bundle, whose base is a Lorentz manifold, (M, g_{ij}) can be endowed with a metrical $d-$connection with torsion for which the conservation laws are verified. We think this is a basic facts to an unitary theory of Finslerian type.

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Faculty of Mathematics
"Al. I. Cuza" University of Iasi

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THE GEOMETRY OF TIME-DEPENDENT LAGRANGIANS

BY

M. ANASTASIEI[2]

Abstract

A generalization of Lagrange geometry appropriate for time-dependent Lagrangians arising in physics and biology, called rheonomic Lagrange geometry, is developed. Nonlinear and linear connections, their torsions, curvatures and deflections are explicitly given. Almost contact structures in rheonomic Lagrange spaces are characterized. Maxwell's equations, for a given Lagrangian, determined by the deflection tensors, are derived.

1 Introduction

Variational principles are basic for most mathematical models in mechanics, physics, ecology, physiology, and so on. These involve Lagrangians or Hamiltonians from which the Euler-Lagrange or Hamilton equations are derived, the theory being then centered on the later. From a geometrical point of view, the most general framework for such a theory is provided by differentiable (smooth) fibre bundles. It means that, for instance, a Lagrangian is a smooth real valued function on the total space TM of the tangent bundle (TM, τ, M) over a smooth manifold M. For a geometrization of such Lagrangians, we refer to [1-5].

There exist certain mathematical models, as for instance those for the three-body problem [6, p. 206] and those concerning ecological systems due to Antonelli [7,8]in which an explicit dependence on time of the Lagrangian (Hamiltonian) is required. A time-dependent Lagrangian is smooth and real valued on $\mathbb{R} \times TM$, where $\mathbb{R}$ is the field of real numbers.

It is our aim to present a geometrization of time-dependent Lagrangians using as a pattern the geometry of Lagrange spaces developed by Miron [3-5,9]. The reader is invited to compare this geometrization to those of [10,11].

[2]I am very indebted to P. L. Antonelli for his interest in this subject. I would like to thank R. Miron for his comments and valuable suggestions during the preparation of this paper.

We begin with some facts (almost tangent structures, nonlinear connections) from the geometry of manifold $\mathbb{R} \times TM$ fibered over $\mathbb{R} \times M$ by $\pi(t, v) = (t, \tau(v))$, $t \in \mathbb{R}$, $v \in TM$. Then we associate with any nonlinear connection N on $E = \mathbb{R} \times TM$ a semispray on E whose integral curves coincide with the paths of N. Regular time-dependent Lagrangians are introduced in Section 3. It is shown that any such Lagrangian L induces a canonical nonlinear connection N_L on E. This nonlinear connection N_L is derived from the Euler-Lagrange equations resulting from a variational problem involving L. Then a metrical almost contact structure on E depending on L only is exhibited. The geometry of L is based on this structure and may be thought of as the counterpart of the almost Kählerian model used in the geometry of time *independent* Lagrangians [3]. As a first step, the linear connections, which are compatible with N_L as well as with the almost contact structure on E, called $N-$linear connections, are studied. The metrical $N-$linear connections are studied too. The existence of a canonical one is shown. Finally, some remarkable time-dependent Lagrangians are considered. Thus, we investigate homogeneous time-dependent Lagrangians.

Similarities with Finsler geometry are emphasized. A second class of time-dependent Lagrangians which we consider contains Lagrangians used in electrodynamics. It is shown that their geometry supports a theory of electromagnetism based on $N-$linear connections.

1.1 On the Geometry of $\mathbb{R} \times TM$

Let M be a smooth manifold of dimension n. It will be assumed Hausdorff connected, and paracompact. We assume $\mathbb{R} \times M$ is coordinated by $(t, x^i) \equiv (t, x)$. The indices $i, j, k, \ldots$, will run over $1, 2, \ldots, n$, and the Einstein summation convention will be used. The coordinates in the fibres of the submersion $\pi : \mathbb{R} \times TM \to \mathbb{R} \times M$ are $(y^i) \equiv (y)$, introduced by $u_{(t,x)} = (t, v_x) = \left(t, y^i \left(\frac{\partial}{\partial x^i}\right)_x\right) \in \pi^{-1}(t, x)$, with $\left(\frac{\partial}{\partial x^i}\right)_x$ the natural basis in the tangent space T_xM, in $x \in M$. Thus, the manifold $\mathbb{R} \times TM$ is coordinated by $(t, x^i, y^i) \equiv (t, x, y)$ and π takes the form $(t, x, y) \to (t, x)$. A change of local coordinates $(t, x, y) \to (\widetilde{t}, \widetilde{x}, \widetilde{y})$ on $E = \mathbb{R} \times TM$ has the following form

$$\widetilde{t} = t, \widetilde{x} = \widetilde{x}^i(x^1, \ldots, x^n),\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^k} y^k, \tag{1.1}$$

with rank $\left(\frac{\partial \widetilde{x}^i}{\partial x^k}\right) = n$.

The natural basis $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ transforms under (1.1) as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t} = \frac{\partial}{\partial \widetilde{t}},\\
&\frac{\partial}{\partial x^j} = \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{\partial}{\partial \widetilde{x}^i} + \frac{\partial \widetilde{y}^i}{\partial x^j}\frac{\partial}{\partial \widetilde{y}^i},\\
&\frac{\partial}{\partial y^j} = \frac{\partial \widetilde{y}^i}{\partial y^j}\frac{\partial}{\partial \widetilde{y}^i}.
\end{aligned}
\tag{2.1}
$$

The kernel of the Jacobian map $D\pi$ supplies a distribution $u \to V_uE$, $u \in E$, on E which will be called the vertical distribution on E. A local basis of the vertical distribution is given by the local vector fields $\left(\frac{\partial}{\partial y^i}\right)$ denoted in what is to follow as $(\dot{\partial}_i)$. From (1.1) and (1.2), it follows that $C = y^i\dot{\partial}_i$ is a global vector field on E. This may be used in order to express the homogeneity with respect to (y^i) of various geometrical objects on E. With the help of (1.2), one may check that setting

$$
J\left(\frac{\partial}{\partial t}\right) = 0, \quad J(\partial_i) = \dot{\partial}_i, J(\dot{\partial}_i) = 0,
\tag{1.3}
$$

where ∂_i stands for $\frac{\partial}{\partial x^i}$ and requiring the linearity of J one obtains a well-defined $(1,1)-$tensor field on E. Moreover, we have $J^2 = 0$, and the Nijenhuis tensor field $N_J(X,Y) = [JX, JY] + J^2[X,Y] - J[JX,Y] - J[X,JY]$, $X, Y \in \chi(E)$, the module of vector fields on E, identically vanishes. Thus J defines an almost tangent structure on E. Sometimes it is convenient to put $t = x^0$ and to use the Greek indices α, β, γ,..., ranging over $0, 1, 2, ..., n$.

A *nonlinear connection* on E is a distribution, called horizontal, $u \to H_uE$, $u \in E$, which is supplementary to the vertical distribution on E. Such a distribution can be given by $(n+1)$ local vector fields, say δ_α. Choosing δ_α such that they are projected by $D\pi$ to $\frac{\partial}{\partial x^\alpha}$ one gets

$$
\delta_\alpha = \partial_\alpha - N^i_\alpha(t,x,y)\dot{\partial}_i,
\tag{1.4}
$$

where ∂_α stands for $\frac{\partial}{\partial x^\alpha}$ and the minus sign is taken for convenience.

The invariance under (1.1) of the horizontal subspaces requires the condition

$$
\delta_\alpha = \frac{\partial \widetilde{x}^\beta}{\partial x^\alpha}\widetilde{\delta}_\beta.
\tag{1.5}
$$

In turn, equation (1.5) implies the following law of transformation for the coefficients N^i_α:

$$
\widetilde{N}^i_\alpha \frac{\partial \widetilde{x}^\alpha}{\partial x^\beta} = \frac{\partial \widetilde{x}^i}{\partial x^k}N^k_\beta - \frac{\partial^2 \widetilde{x}^i}{\partial x^\beta \partial x^k}y^k.
\tag{1.6}
$$

If one rewrites (1.4) in the form

$$\delta_0 = \frac{\partial}{\partial t} - N_0^i(t,x,y)\dot{\partial}_i,\ \delta_i = \partial_i - N_i^k(t,x,y)\dot{\partial}_k, \tag{1.7}$$

one may state the following theorem.

Theorem 1.1. *To give a nonlinear connection on E is equivalent to giving a set of functions (N_0^k, N_i^k) defined in each coordinate chart on E, which transform under* (1.1) *as follows:*

$$\widetilde{N}_0^k(\widetilde{t},\widetilde{x},\widetilde{y}) = \frac{\partial \widetilde{x}^k}{\partial x^h} N_0^h(t,x,y), \tag{1.8}$$

$$\widetilde{N}_i^k(\widetilde{t},\widetilde{x},\widetilde{y})\frac{\partial \widetilde{x}^i}{\partial x^j} = \frac{\partial \widetilde{x}^k}{\partial x^i} N_j^i(t,x,y) - \frac{\partial \widetilde{y}^y}{\partial x^j}. \tag{1.9}$$

Proof. If a nonlinear connection on E is given by the local coefficients (N_α^i) satisfying (1.6), taking into account (1.5) and (1.7), it comes out that equation (1.6) is equivalent to (1.8) and (1.9). Conversely, a set of functions defined in each coordinate chart on E verifying (1.6) on overlaps, provides, according to (1.4) and (1.5), a nonlinear connection on E. ■

The local coefficients $(N_0^i(t,x,y))$ transform under (1.1) like the components of a vector field on M, although they depend on t, x and y. We shall say (N_0^i) define a distinguished vector field on E, briefly a $d-$vector field. More generally, an $(r,s)-$tensor field on E whose local components transform like those of an $(r,s)-$tensor field on M, ignoring their dependence on t, x, and y, will be called a $d-$tensor field of type (r,s). A similar situation appears in [12], where a d-tensor field is called a Finsler tensor field, as well as in [7, p. 131], where a $d-$tensor field is called a Douglas tensor field.

The local coefficients $(N_i^k(t,x,y))$ transform under (1.1) like those of a nonlinear connection on TM [3]. When these local coefficients do not depend on t, they really define a nonlinear connection on TM. Conversely, a nonlinear connection on TM paired with a $d-$vector field on E defines a nonlinear connection on E.

The decomposition $T_uE = H_uE \oplus V_uE$ gives rise to two projectors, an horizontal one denoted by h and a vertical one denoted by v, as well as to an almost product structure $P = h - v$. All these depend smoothly on $u \in E$ and thus induce $(1,1)-$tensor fields on E, which will be denoted again by h, v, and P, respectively.

There exist many ways for introducing the curvature of a nonlinear connection. We choose the following formal one since it allows us to relate quickly the curvature to the integrability of the horizontal distribution. Namely, the *curvature* Ω of a nonlinear connection is defined as the Nijenhuis tensor field N_h of the horizontal projector h, that is $\Omega = N_h$. In a coordinate chart Ω, it is given as follows:

$$\Omega(\delta_\alpha, \delta_\beta) = R^i{}_{\alpha\beta}\dot{\partial}_i,\ \Omega(\partial_\alpha, \dot{\partial}_i) = 0,\ \Omega(\dot{\partial}_i, \dot{\partial}_j) = 0, \tag{1.10}$$

$$R^i_{\alpha\beta} = \delta_\beta N_\alpha^i - \delta_\alpha N_\beta^i = \partial_\beta N_\alpha^i - \partial_\alpha N_\beta^i + N_\alpha^k \dot{\partial}_k N_\beta^i - N_\beta^k \dot{\partial}_k N_\alpha^i. \tag{1.11}$$

On the other hand, we have

$$[\delta_\alpha, \delta_\beta] = R^i_{\alpha\beta}\dot{\partial}_i, \ [\delta_\alpha, \dot{\partial}_i] = \dot{\partial}_i N^i_\alpha \dot{\partial}_j. \tag{1.12}$$

Thus, the horizontal distribution on E is integrable if and only if $\Omega = 0$, or equivalently, $R^i_{\alpha\beta} = 0$. We notice that R^i_{jk} and R^i_{ok} define $d-$tensor fields on E of type $(1,2)$ and $(1,1)$, respectively.

Let $B^k_{j\alpha} = \dot{\partial}_j N^k_\alpha$. Differentiating with respect to y^j both sides (1.8), one finds that $B^k{}_{j0}$ defines a $d-$tensor field of type $(1,1)$. Proceeding similarly with (1.9), one gets

$$\widetilde{B}^k{}_{rs}\frac{\partial \widetilde{x}^r}{\partial x^j}\frac{\partial \widetilde{x}^s}{\partial x^i} = \frac{\partial \widetilde{x}^k}{\partial x^h}B^h{}_{ji} - \frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}. \tag{1.13}$$

Thus, the functions $B^k{}_{ji}(t,x,y)$ transforms under (1.1) as the local coefficients of a classical linear connection, although they depend on t, x, y. It is said in [7, p. 131] that $B^h{}_{ji}$ define a Douglas connection. We have used the letter B since in [12] these functions are related to the so-called Berwald connection.

A nonlinear connection $N(N^i_\alpha)$ is homogeneous (resp. linear) if the functions $N^i_0(t,x,y)$ and $N^i_k(t,x,y)$ are homogeneous of degree one (resp. linear) with respect to (y^i). Of course, in order to speak about homogeneous connections we must delete from E the points $(t,x,0)$ because any homogeneous real function of class C^1 at the origin becomes linear.

When a linear connection (N^i_0, N^i_j) is given, the equalities $N^i_0(t,x,y) = K^i_j(t,x)y^j$, $N^i_j(t,x,y) = \Gamma^i_{jk}(t,x)y^k$ provide a pair $(K^i_j(t,x), \Gamma^i_{jk}(t,x))$ which may be thought of as a general affine connection on $\mathbb{R} \times M$ in the sense of [13].

2 Semisprays and nonlinear connections

A *time-dependent vector field* on TM is a smooth map $X^0 : \mathbb{R} \times TM \to T(TM)$, $(t,u) \to X^0(t,u) \in T_u(TM)$, $u \in TM$. It induces a vector field X on $\mathbb{R}\times TM$ by setting $X(t,u) = (1, X^0(t,u))$, and we have also $X = \dfrac{\partial}{\partial t} + X^0$ [2].

A *time-dependent semispray* (second order differential equation on M) is a time-dependent vector field S^0 on TM which satisfies

$$D\tau \circ S^0(u) = u, \quad \text{for all } u \in TM. \tag{2.1}$$

The vector field S induced on $\mathbb{R} \times TM$ by a time-dependent semispray S^0, that is

$$S = \frac{\partial}{\partial t} + S^0, \tag{2.2}$$

will be called a *semispray*.

According to (2.1) and (2.2) a semispray in a coordinate chart on E takes the form

$$S = \frac{\partial}{\partial t} + y^i \partial_i + S^i(t, x, y)\dot{\partial}_i, \tag{2.3}$$

with (S^i) verifying on overlaps

$$\widetilde{S}^i = \frac{\partial \widetilde{x}^i}{\partial x^k} S^k + \frac{\partial^2 \widetilde{x}^i}{\partial x^j \partial x^k} y^j y^k. \tag{2.4}$$

Conversely, a vector field S which in each coordinate chart has the form (2.3) such that equation (2.4) is fulfilled, is a semispray.

A direct calculation gives the following result.

Proposition 2.1. *A vector field S on E is a semispray if and only if $dt(S) = 1$, $\psi^i(S) = 0$, with $\psi^i = dx^i - y^i dt$.*

A relationship between semisprays and nonlinear connections is given by the following two theorems.

Theorem 2.1. *Let N be a nonlinear connection given by the local coefficients $(N_0^i(t, x, y), N_k^i(t, x, y))$. Then $S = \dfrac{\partial}{\partial t} + y^i \partial_i - (N_0^i + N_k^i y^k)\dot{\partial}_i$ is a semispray.*

Theorem 2.2. *Let $S(S^i)$ be a semispray. Then $\left(\dfrac{\partial S^i}{\partial t}, -\dfrac{1}{2}\dfrac{\partial S^i}{\partial y^j}\right)$ are local coefficients for a nonlinear connection on E.*

The proofs follow by showing that equations (1.8) and (1.9) imply (2.3), and conversely.

A time-dependent semispray is said to be a spray if it is invariant under (a gauge transformation, dilatation or contraction) *similarity* on TM and a semispray will be called a spray if it is provided by a time dependent spray S^0. It is immediate that a semispray is a spray if and only if the functions $(S^i(t, x, y))$ are homogeneous of degree 2 in (y^i). The later condition is clearly compatible with (2.4).

If $S(S^i)$ is a spray, then $(0, -\dfrac{1}{2}\dot{\partial}_j S^i)$ are the local coefficients of a homogeneous connection. Conversely, a homogeneous connection defines a spray $S^i = -N_k^i y^k$.

Let $c : \mathbb{R} \to M$ be a smooth curve on M and $\dot{c} : \mathbb{R} \to TM$ its tangent vector field. Then $\sigma(t) = (t, \dot{c}(t))$ defines a smooth curve on $\mathbb{R} \times TM$. We say this curve is an integral curve of a semispray S if

$$\dot{\sigma}(t) = S(\sigma(t)), \quad t \in \mathbb{R}. \tag{2.5}$$

If we assume that $c(t)$ belongs to a coordinate chart for all $t \in \mathbb{R}$, and we take $x^i = x^i(t)$, $t \in \mathbb{R}$, as the equations of the curve c, then equation (2.5) is equivalent to

$$\frac{d^2 x^i}{dt^2} = S^i(t, x, \dot{x}), \quad \dot{x} = \frac{dx}{dt}, \tag{2.6}$$

because of $\dot{\sigma}(t) = \dfrac{\partial}{\partial t} + \dfrac{dx^i}{dt}\dot{\partial}_i + \dfrac{d^2 x^i}{dt^2}\dot{\partial}_i$.

A curve $c : t \to c(t)$, $t \in \mathbb{R}$, on M is said to be a path for a nonlinear connection N if the curve $\sigma : t \to (t, \dot{c}(t))$ is horizontal with respect to N, that is, its tangent vector field belongs to the horizontal distribution on E.

In a coordinate chart containing $\sigma(t)$, $t \in \mathbb{R}$, if $(\delta_\alpha, \dot{\partial}_i)$ is the adapted basis introduced before and $(dx^\alpha, \delta y^i)$, with $\delta y^i = dy^i + N^i_\alpha dx^\alpha$ is its dual, it appears as obvious that σ is an horizontal curve if and only if $\delta y^i(\dot{\sigma}) = 0$ for every $i = 1, ..., n$. Writing down these equations, one obtains the following theorem.

Theorem 2.3. *A curve $c : t \to x^i(t)$, $t \in \mathbb{R}$, on M is a path for a nonlinear connection (N^i_0, N^i_j) if and only if*

$$\frac{d^2x^i}{dt} + N^i_j(t, x, \dot{x})\frac{dx^j}{dt} + N^i_0(t, x, \dot{x}) = 0, \quad x^i = \frac{dx^i}{dt}. \tag{2.7}$$

Looking at the semispray associated with a nonlinear connection one immediately gets the next result.

Theorem 2.4. *The paths of a nonlinear connection coincide with the integral curves of the semispray associated with it.*

We notice that the systems of differential equations (2.6) and (2.7) do not remain in the same form if an arbitrary change of parameter is performed. They keep their form only if one sets $\tilde{t} = \pm t + a$, with $a \in \mathbb{R}$. Thus, t plays the role of an affine parameter. We conclude that the solutions of these systems have to be considered together with the parameters in which they are given. In other words, the curve c in the above has to be thought of as a parameterized curve.

3 Time-dependent lagrangians

Now, we shall point out that a regular time-dependent Lagrangian defines a nonlinear connection on $E = \mathbb{R} \times TM$ and, thus, a semispray on E.

A smooth function $L : \mathbb{R} \times TM \to \mathbb{R}$, $(t, v) \to L(t, v)$, is called a *time-dependent* Lagrangian on M. It is said L is *regular* if the matrix with the entries

$$g_{ij}(t, x, y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}, \tag{3.1}$$

is of rank n on E.

The condition for L regular does not depend on the coordinate chart involved.

Definition 3.1. A pair $RL^n = (M, L(t, x, y))$ in which L is a regular time-dependent Lagrangian such that the quadratic form with the coefficients g_{ij} from (3.1) has constant signature, will be called a *rheonomic Lagrange space.*

Let $c : t \to c(t)$, $t \in \mathbb{R}$ be a parameterized curve on M as before. If its image is in a coordinate chart, one may take $x^i = x^i(t)$, $t \in \mathbb{R}$ as its local representation, and then its tangent vector field $\dot{c}$ is locally represented as $(x^i(t), \dot{x}^i(t))$. When a regular time dependent Lagrangian L on M is given,

one may define a functional

$$\mathcal{L} : c \to \mathcal{L}(c) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt,$$

which suggests the following variational problem: find those curves, called extremals, which afford extremal values for $\mathcal{L}$. Looking for such an extremal in the space of all curves with fixed end points, one finds [1, p. 153; 8, p. 58], that it is among curves which are solutions of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0. \tag{3.2}$$

Now, if one considers the curve $\widetilde{c} = (t, c(t))$, $t_0 \leq t \leq t_1$, on $\mathbb{R} \times M$, it comes out [8, p. 58] that $\widetilde{c}$ is an extremal of the functional

$$\mathcal{L}(\widetilde{c}) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt,$$

on the space of curves joining (t_0, x_0) and (t_1, x_1) if the Euler-Lagrange equations are satisfied along $\widetilde{c}$.

Expanding the derivative with respect to t, the equations (3.2) can be put in the form

$$2g_{ij}\frac{d^2x^j}{dt^2} + \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\dot{x}^j - \frac{\partial L}{\partial x^i}\right) + \frac{\partial^2 L}{\partial t \partial \dot{x}^i} = 0. \tag{3.3}$$

Using the inverse (g^{ki}) of the matrix (g_{ij}), one resolves (3.3) with respect to $\dfrac{d^2x^k}{dt^2}$ as follows:

$$\frac{d^2x^k}{dt^2} + 2G^k(t, x, \dot{x}) + N_0^k(t, x, \dot{x}) = 0, \tag{3.4}$$

in which the following notations were used

$$N_0^k(t, x, y) = \frac{1}{2}g^{ki}\frac{\partial^2 L}{\partial t \partial y^i}, \tag{3.5}$$

$$G^k(t, x, y) = \frac{1}{4}g^{ki}\left(\frac{\partial^2 L}{\partial y^i \partial x^j}y^j - \frac{\partial L}{\partial x^i}\right), \quad y^i = \dot{x}^i. \tag{3.6}$$

Now, we state the following result:

Theorem 3.1. *The functions* $N_L = (N_0^k(t, x, y),\ N_i^k(t, x, y))$, *where* N_0^k *is given by* (3.5) *and* N_i^k *by*

$$N_i^k(t, x, y) = \frac{\partial G^k(t, x, y)}{\partial y^i}, \tag{3.7}$$

are local coefficients of a nonlinear connection on E completely determined by L.

Proof. Under the coordinate transformation $(t,x,y) \to (\widetilde{t},\widetilde{x},\widetilde{y})$, given by (1.1), $\dfrac{\partial L}{\partial t}$ is invariant; $\dfrac{\partial L}{\partial y^i}$ transform like a covector on M, i.e., they define a $d-$covector field, and because (g^{kj}) transforms like the components of a $d-$tensor field of type $(2,0)$, it follows that (N_0^k) from (3.5) are the components of a $d-$vector field on E. The partial derivatives of L take, under (1.1), the following form:

$$\frac{\partial L}{\partial x^i} = \frac{\partial L}{\partial \widetilde{x}^k}\frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial L}{\partial \widetilde{y}^k}\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}y^j,$$

$$\frac{\partial^2 L}{\partial y^i \partial x^k} = \frac{\partial^2 L}{\partial \widetilde{y}^j \partial \widetilde{y}^h}\frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial \widetilde{x}^h}{\partial x^k} + \frac{\partial L}{\partial \widetilde{y}^j}\frac{\partial^2 \widetilde{x}^j}{\partial x^i \partial x^k} + 2\widetilde{g}_{jh}\frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial^2 \widetilde{x}^h}{\partial x^k \partial x^s}y^s.$$

Multiplying the second equality by y^k, we introduce the result in the form of (3.6):

$$4g_{ij}G^j = \frac{\partial^2 L}{\partial y^i \partial x^k}y^k - \frac{\partial L}{\partial x^i},$$

which thus become

$$4g_{ij}G^j = \left(\frac{\partial^2 L}{\partial \widetilde{y}^j \partial \widetilde{x}^k}\widetilde{y}^k - \frac{\partial L}{\partial \widetilde{x}^j}\right)\frac{\partial \widetilde{x}^j}{\partial x^i} + 2g_{jk}\frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial^2 \widetilde{x}^k}{\partial x^r \partial x^s}y^r y^s,$$

since two terms cancel each other. Hence, we obtain the transformation law of G^i as follows:

$$\frac{\partial \widetilde{x}^k}{\partial x^i}G^i = \widetilde{G}^k + \frac{1}{2}\frac{\partial^2 \widetilde{x}^k}{\partial x^i \partial x^j}y^i y^j,$$

on account of $\operatorname{rank}(g_{jj}) = \operatorname{rank}\left(\dfrac{x^j}{x^i}\right) = n$. Differentiating both sides of the last equality with respect to y^j, one gets N_k^i, from equation (3.7) has the transformation law (1.9), and the proof is complete. ∎

As we have seen in Section 2, N_L defines two semisprays on E given by

$$S_1^i = -\frac{1}{2}g^{ik}\frac{\partial^2 L}{\partial t \partial y^k} - \frac{\partial G^i}{\partial y^k}y^k \quad \text{and} \quad S_0^i = -\frac{\partial G^i}{\partial y^k}y^k,$$

respectively. They coincide if, for instance, $\dfrac{\partial L}{\partial y^k}$ do not depend on t. Note that these semisprays are determined by L only.

Remark 3.1. The nonlinear connection N_L is without torsion. Let us consider an 1-form $\omega = \dfrac{\partial L}{\partial y^i}dx^i + \left(L - \dfrac{\partial L}{\partial y^i}y^i\right)dt$ on E and let

$$\theta = d\omega = \left[d\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i}dt\right] \wedge (dx^i - y^i dt).$$

Thus, θ defines a contact structure on E. A vector field X on E is said to be characteristic for θ if the inner product of θ by X vanishes, that is $X \cdot \theta = 0$.

A curve on E is said to be characteristic for θ if its tangent vector field is characteristic for θ. We get the following theorem.

Theorem 3.2. *If a curve $c : t \to c(t)$ on M is an extremal for $\mathcal{L}$, then the curve $\sigma(t) = (t, \dot{c}(t))$ is a characteristic curve for θ.*

Proof. As we have seen before, $\theta = \varphi_i \wedge \psi^i$, where $\varphi_i = d\left(\frac{\partial L}{\partial y^i}\right) \frac{\partial L}{\partial x^i} dt$ and $\psi^i = dx^i - y^i dt$.

Next, $(\dot{\sigma}(t) \cdot \theta)(Y) = \varphi_i(\dot{\sigma}(t))\psi^i(Y) - \varphi_i(Y)\psi^i(\dot{\sigma}(t))$ for any $Y \in \chi(E)$. But $\psi^(\dot{\sigma}(t)) = 0$, since along σ, $y^i = \frac{dx^i}{dt}$ and $\varphi_i(\dot{\sigma}(t)) = 0$ by virtue of the Euler-Lagrange eqs. Thus, $\dot{\sigma}(t) \cdot \theta = 0$. ∎

This theorem opens up a way in which contact geometry can come into the theory of Lagrangian systems [2]. We do not follow this way. Our geometrization is centered on a metrical structure derived from a regular time-dependent Lagrangian.

Finally, if we compare (3.4) with (2.7), we see that if G^k is homogeneous of degree two with respect to y, a fact which is equivalent to $N^k_i y^i = 2G^k$, it follows that the extremals of $\mathcal{L}$ coincide with the paths of the canonical nonlinear connection N_L and with the integral curves of the semispray associated with N_L as well.

4 A metrical almost contact structure on E

Let $R^n = (M, L(t,x,y))$ be a rheonomic Lagrange space. The canonical nonlinear connection produces a decomposition of the tangent bundle TE as a direct sum $TE = HE \oplus VE$. Let $(\delta_0, \delta_i, \dot{\partial}_i)$ be the local frame adapted to this decomposition and $(dt, dx^i, \delta y^i)$ its dual. Let us consider a linear map $F : T_u E \to T_u E$ given by

$$F(\delta_0) = 0, \quad F(\delta_i) = -\partial_i, \quad F(\dot{\partial}_i) = \delta_i. \tag{4.1}$$

Then $u \to F_u$, $u \in E$, defines an $(1,1)-$tensor field on E. It is obvious that $\mathrm{rank} F = 2n$ and an easy calculation gives $F^3 + F = 0$. Thus F defines an $f(3,1)-$structure on E [11].

Theorem 4.1. *Let $RL^n = (M, L(t,x,y))$ be a rheonomic Lagrange space. Then the manifold $E = \mathbb{R} \times TM$ carries an almost contact structure (F, δ_0, dt).*

Proof. We have $dt(\delta_0) = 1$ and equation (4.1) gives $F^2(\delta_i) = -\delta_i$, $F^2(\dot{\partial}_i) = -\dot{\partial}_i$. Thus it follows that $F^2 = -I + \delta_0 \times dt$.

The torsion tensor field [14] of the almost contact structure (F, δ_0, dt) reduces to the Nijenhuis tensor N_F of F. Thus, the almost contact structure (F, δ_0, dt) is normal if and only if $N_F = 0$.

Evaluating N_F in the frame (δ_0, δ_i, dt) one obtains the following theorem.

Theorem 4.2. *The almost contact structure (F, δ_0, dt) is normal if and only if*

(1) *The canonical nonlinear connection* $N_L = (N_0^k, N_i^k)$ *is without curvature; and*

(2) $\dot{\partial}_i N_0^k = 0$.

As it is easy to check, the functions (g_{ij}) given by (3.1) are the components of a $d-$tensor of type $(0,2)$ on E. This will be called the metrical or fundamental tensor field of RL^n. Using it we can define the following $(0,2)-$tensor field on E:

$$G = dt \otimes dt + g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \tag{4.2}$$

We notice that if (g_{ij}) is positive definite, then G is a Riemann metric on E. Otherwise, it is said it defines a metrical structure on E.

Remark 4.1. The horizontal and vertical distributions are orthogonal each to the other with respect to G.

A direct calculation in the frame $(\delta_0, \delta_i, \dot{\partial}_i)$ gives the following result.

Theorem 4.3. *The metrical structure* G *satisfies the following equations*

$$\begin{aligned} &G(FX, FY) = G(X,Y) - dt(X)dt(Y), \\ &dt(X) = G(\delta_0, X), \quad X, Y \in \chi(E). \end{aligned} \tag{4.3}$$

In other words, the previous theorem says that (F, δ_0, dt, G) is a metrical almost contact structure on E. Recall that this metrical almost contact structure is completely determined by L. The particular form of L could provide examples of structures which cover the classifications quoted in [11].

5 $N-$linear connections on E

Now we shall consider a class of linear connections on E which are compatible with a nonlinear connection N, in particular with N_L, as well as with the almost contact structure associated with it. These will be called $N-$linear connections, recalling their compatibility with N.

The decomposition $T_uE = H_uE \oplus V_uE$ produced by a nonlinear connection N induces a decomposition

$$X = X^H + X^V, \quad X \in \chi(E), \tag{5.1}$$

where $X^H(X^V)$, is a vector field on E taking its values in horizontal (vertical) distribution.

The decomposition (5.1) induces a decomposition of any tensor field on E in horizontal and vertical parts. We denote also by h and v the horizontal and vertical projectors defined by (5.1), and then $P = h - v$ is an almost product structure on E.

Definition 5.1. A linear connection $D : \chi(E) \times \chi(E) \to \chi(E)$, $(X, Y) \to D_XY$ is said to be an $N-$linear connection if

$$(a) \;\; D_XP = 0, \;\; (b) \;\; D_XF = 0, \;\; (c) \;\; D_X\delta_0 = 0,$$

hold for any $X \in \chi(E)$.

Condition (a) is equivalent to the fact that D_X preserves by parallelism the horizontal and vertical distributions, i.e., $(D_X Y^H)^V = 0$ and $(D_X Y^V)^H = 0$, or $D_X Y = (D_X Y^H)^H + (D_X Y^V)^V$. Now, if one sets

$$D_X^H Y = D_{X^H} Y, \quad D_X^H f = X^H f, \quad f \in \mathcal{F}(E) \tag{5.2}$$

and extends D_X^H to any $d-$tensor field on E by the usual method, one obtains an operator called the $h-$covariant derivation in the algebra of $d-$tensor fields on E. Similarly, one may construct an operator for the $v-$covariant derivation, setting

$$D_X^V Y = D_{X^V} Y, \quad D_{X^V} f = X^V f, \quad f \in \mathcal{F}(E). \tag{5.3}$$

Now we state the following local characterization of an $N-$linear connection.

Theorem 5.1. *To give an $N-$linear connection on E is equivalent to give in every local chart on E, a set of functions $D\Gamma = (L^i_{j0}, L^i_{jk}, C^i_{jk})$ which satisfy on overlaps,*

$$\begin{aligned} \widetilde{L}^i_{j0} \frac{\partial \widetilde{x}^j}{\partial x^h} &= \frac{\partial \widetilde{x}^i}{\partial x^k} L^k_{h0}, \\ \widetilde{L}^i_{jk} \frac{\partial \widetilde{x}^j}{\partial x^r} \frac{\partial \widetilde{x}^k}{\partial x^s} &= \frac{\partial \widetilde{x}^i}{\partial x^h} L^h_{rs} - \frac{\partial^2 \widetilde{x}^i}{\partial x^r \partial x^s}, \\ C^i_{jk} \frac{\partial \widetilde{x}^j}{\partial x^r} \frac{\partial \widetilde{x}^k}{\partial x^s} &= \frac{\partial \widetilde{x}^i}{\partial x^h} C^h_{rs}. \end{aligned} \tag{5.4}$$

Proof. If we express $D_X Y$ in a local chart, it comes out that it is well-defined by

$$D_{\delta_0} \delta_j = L^0_{j0} \delta_0 + L^i_{j0} \delta_i, \quad D_{\delta_0} \dot{\partial}_j = M^i_{j0} \dot{\partial}_i,$$

$$D_{\delta_k} \delta_j = L^0_{jk} \delta_0 + L^i_{jk} \delta_i, \quad D_{\delta_k} \dot{\partial}_j = M^i_{jk} \dot{\partial}_i,$$

$$D_{\dot{\partial}_k} \delta_j = Q^0_{jk} \delta_0 + Q^i_{jk} \delta_i, \quad D_{\dot{\partial}_k} \dot{\partial}_j = C^i_{jk} \dot{\partial}_i,$$

where (a) and (c) from Definition 5.1 were taken into consideration. Taking into consideration (b) from the same definition, these equations reduce to

$$\begin{aligned} &D_{\delta_0} \delta_j = L^i_{j0} \delta_i, \quad D_{\delta_k} \delta_j = L^i_{jk} \delta_i, \quad D_{\dot{\partial}_k} \delta_j = C^i_{jk} \delta_i, \\ &D_{\delta_0} \dot{\partial}_j = L^i_{j0} \dot{\partial}_i, \quad D_{\delta_k} \dot{\partial}_j = L^i_{jk} \delta_i, \quad D_{\dot{\partial}_k} \dot{\partial}_j = C^i_{jk} \dot{\partial}_i, \end{aligned} \tag{5.5}$$

and thus a set of functions $D\Gamma = (L^i_{j0}, L^i_{jk}, C^i_{jk})$ appears. If a transformation of coordinates on E is performed, it turns out that these functions satisfy (5.4).

Conversely, given a set of functions $D\Gamma$, which on overlaps satisfy (5.4), by using (5.5), a well-defined linear connection on E is obtained, and by a

direct calculation one proves it satisfies (a), (b), and (c) from Definition 5.1, that is, it is an $N-$linear connection. ■

We notice that (5.4) shows that (L^i_{j0}) are the components of a $d-$tensor field of type $(1,1)$, (C^i_{jk}) are the components of a $d-$tensor field of type $(1,2)$ and (L^i_{jk}) define a Douglas connection.

Let

$$T = t^{0\dots i\dots}_{\dots j\dots}\delta_0 \otimes \dots \times \delta_i \otimes \dots \times \delta y^j \otimes \dots$$

be a $d-$tensor field on E. By (5.5), one obtains the $h-$ and $w-$covariant derivative of T as follows:

$$D_{\delta_0}T = t^{0\dots i\dots}_{\dots j\dots|0}\delta_0 \otimes \dots \otimes \delta_i \times \dots \otimes \delta y^j \otimes \dots,$$

$$D^H_X T = X^k t^{0\dots i\dots}_{\dots j\dots|k}\delta_0 \otimes \dots \times \delta_i \otimes \dots \times \delta y^j \otimes \dots,$$

$$D^V_X T = \dot{X}^k t^{0\dots i\dots}_{\dots j\dots|k}\delta_0 \otimes \dots \times \delta_i \otimes \dots \times \delta y^j \otimes \dots,$$

when $X = X^k\delta_k + \dot{X}^k\dot{\partial}_k$, where we have put

$$t^{0\dots i\dots}_{\dots j\dots|0} = \delta_0 t^{0\dots i\dots}_{\dots j\dots} + L^i_{k0}t^{0\dots k\dots}_{\dots j\dots} - L^k_{j0}t^{0\dots i\dots}_{\dots k\dots},$$

(5.6) $$t^{0\dots i\dots}_{\dots j\dots|h} = \delta_h t^{0\dots i\dots}_{\dots j\dots} + L^i_{kh}t^{0\dots k\dots}_{\dots j\dots} - L^k_{jh}t^{0\dots i\dots}_{\dots k\dots},$$

$$t^{0\dots i\dots}_{\dots j\dots|h} = \dot{\partial}_h t^{0\dots i\dots}_{\dots j\dots} + L^i_{kh}t^{0\dots k\dots}_{\dots j\dots} - C^k_{jh}t^{0\dots i\dots}_{\dots k\dots},$$

The torsion of an $N-$linear connection decomposes because of (5.1) into five $d-$tensor fields (the sixth identically vanishes) whose local components, in the adapted frame, are the following:

(5.6′)
$$T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad R^i_{\alpha\beta} = \delta_\beta N^i_\alpha - \delta_\alpha N^i_\alpha, \quad C^i_{jk},$$
$$P^i_{0j} = \dot{\partial}_j N^i_0 - L^i_{j0}, \quad P^i_{kj} = \dot{\partial}_j N^i_k - L^i_{jk}, \qquad S^i_{jk} = C^i_{jk} - C^i_{kj}.$$

All these functions will be called torsions of $D\Gamma$.

Analogously, one may prove that the curvature of an $N-$linear connection is locally determined by the following functions called curvatures of $D\Gamma$:

$$R_j{}^i{}_{\alpha\beta} = \delta_\beta L^i{}_{j\alpha} - \delta_\alpha L^i{}_{j\beta} + L^i_{h\beta}L^h_{j\alpha} - L^h_{j\beta}L^i_{h\alpha} + C^i{}_{jh}R^h_{\alpha\beta},$$

$$P_j{}^i{}_{0k} = \dot{\partial}_k L^i{}_{j0} - C^i{}_{jk|0} + C^i{}_{jh}P^h{}_{0k},$$

(5.7)

$$P_j{}^i{}_{hk} = \dot{\partial}L^i{}_{jh} - C^i{}_{jh|k} + C^i{}_{js}P^s{}_{hk},$$

$$S_j{}^i{}_{hk} = \dot{\partial}_k C^i{}_{jh} - \dot{\partial}_h C^i{}_{jk} + C^s{}_{jh}C^i{}_{sk} - C^s{}_{jk}C^i{}_{sh}.$$

We say T^i_{jk} is the $h(hh)-$torsion of $D\Gamma$ and $S^i{}_{jk}$ is $v(vv)-$torsion of $D\Gamma$. The torsions and curvatures of $D\Gamma$ satisfy a number of a Bianchi identities. We do not write them here.

6 Metrical $N-$linear connections

Let $RL^n = (M, L)$ be a rheonomic Lagrange space and let us consider $N-$linear connections on E which are compatible with the metrical structure G defined by L.

Definition 6.1. An $N-$linear connection D on E is said to be metrical if $D_X g = 0$, for any $X \in \chi(E)$.

A direct calculation in local coordinates leads to the following result.

Theorem 6.1. *An $N-$linear connection $D = (L^i_{j0}, L^i_{jk}, C^i_{jk})$ is metrical if and only if*

$$g_{ij|0} = 0, \quad g_{ij|k} = 0, \quad g_{ij}|_k = 0. \tag{6.1}$$

As to the existence of metrical $N-$linear connections, we have the following theorem.

Theorem 6.2. *Let $RL^n = (M, L(t, x, y))$ be a rheonomic Lagrange space. If ${T^i}_{jk}$ and ${S^i}_{jk}$ are two arbitrary skew-symmetric $d-$tensor fields on $E = \mathbb{R} \times TM$, then there exists a set of metrical $N-$linear connections on E, such that each of them has ${T^i}_{jk}$ and ${S^i}_{jk}$ as $h(hh)-$ and $v(vv)-$torsions; respectively. The local coefficients of a connection from this set are given as follows:*

$$\begin{aligned}
{L^k}_{i0} &= \frac{1}{2} g^{kh} \delta_0 g_{ih} + O^{jk}_{ih} X^h_{j0},\\
{L^k}_{ij} &= \frac{1}{2} g^{kh} (\delta_i g_{hj} + \delta_j g_{ih} - \delta_h g_{ij} + g_{is} {T^s}_{jk} + g_{js} {T^s}_{ih} + g_{hs} {T^s}_{ij}),\\
{C^k}_{ij} &= \frac{1}{2} g^{kh} (\dot{\partial}_i g_{hj} + \dot{\partial}_j g_{ih} - \dot{\partial}_h g_{ij} + g_{is} {S^s}_{jk} + g_{js} {S^s}_{ih} + g_{hs} {S^s}_{ij}),
\end{aligned} \tag{6.2}$$

where ${X^h}_{j0}$ is an arbitrary $d-$tensor field on E, and O^{jk}_{ih} denotes the Obata operator

$$O^{jk}_{ih} = \frac{1}{2} (\delta^j_i \delta^k_h - g_{ih} g^{jk}). \tag{6.3}$$

Proof. The condition $g_{ij|k} = 0$ is equivalent to $\delta_k g_{ij} = {L^h}_{ik} g_{hj} + {L^h}_{jk} g_{ih}$. Permuting (k, i, j) to (i, j, k) and (j, k, i) in this equality, adding two and subtracting one from the equalities thus obtained, and denoting ${L^i}_{jk} - {L^i}_{kj} = {T^i}_{jk}$, one obtains ${L^k}_{ij}$ in the form (6.2). One may proceed analogously in order to obtain ${C^k}_{ij}$ as in (6.2), using $g_{ij}|_k = 0$ and denoting ${C^i}_{jk} - {C^i}_{kj} = {S^i}_{jk}$.

Next, it is easy to check that ${L^k}_{i0} = \dfrac{1}{2} g^{kh} \delta_0 g_{ih}$ are solutions of the equations $g_{ij|0} = \delta_0 g_{ij} - {L^k}_{i0} g_{kj} - {L^k}_{j0} g_{ik} = 0$, in the unknowns ${L^k}_{i0}$.

Now, if ${L^k}_{i0}$ are any solutions of these equations, then ${B^k}_{i0} = {L^k}_{i0} - \dfrac{1}{2} g^{kh} \delta_0 g_{ih}$ satisfy the equations $g_{ki} {B^k}_{j0} + g_{jk} {B^k}_{i0} = 0$. The general solutions

of these equations are $B^k{}_{j0} = O^{jk}_{ih} X^h_{j0}$, where X^h_{j0} is an arbitrary $d-$tensor field. Thus $L^k{}_{i0}$ has the form given in (6.2). ■

Taking $X^i{}_{j0} = 0$ in (6.2) one obtains the following corollary.

Corollary 6.1. *Let $RL^n = (M, L(t,x,y))$ be a rheonomic Lagrange space and $T^i{}_{jk}$, $S^i{}_{jk}$ be two arbitrary skew-symmetric $d-$tensor fields on E. Then, there exists an unique metrical $N-$linear connection $D\Gamma = \left(\frac{1}{2} g^{kh} \delta_0 g_{ih}, L^k{}_{ij}, C^k{}_{ij}\right)$, whose $h(hh)-$torsion is $T^i{}_{jk}$ and $v(vv)-$torsion is $S^i{}_{jk}$. The coefficients $L^i{}_{jk}$ and $C^i{}_{jk}$ are given by (6.2).*

Proof. The uniqueness of $L^i{}_{jk}$ and $C^i{}_{jk}$ from (6.2) follows by contradiction. ■

In particular, taking $T^i{}_{jk} = S^i{}_{jk} = 0$, one obtains the next corollary.

Corollary 6.2. *Let $RL^n = (M, L(t,x,y))$ be a rheonomic Lagrange space. Then, there exists a set of metrical $N-$linear connections, such that each of them has the vanishing $h(hh)-$ and $v(vv)-$torsion. The local coefficients of any connection of this set are given by*

$$\begin{aligned} L^k{}_{i0} &= \frac{1}{2} g^{kh} \delta_0 g_{hi} + O^{jk}_{ih} X^h{}_{j0}, \\ L^k{}_{ij} &= \frac{1}{2} g^{kh} (\delta_i g_{hj} + \delta_j g_{hi} - \delta_h g_{ij}), \\ C^k{}_{ij} &= \frac{1}{2} g^{kh} (\dot{\partial}_i g_{hj} + \dot{\partial}_j g_{hi} - \dot{\partial}_h g_{ij}), \end{aligned} \tag{6.4}$$

where $X^h{}_{j0}$ is an arbitrary $d-$tensor field on E.

Definition 6.2. The metrical $N-$linear connection whose local coefficients are given by (6.4), with $X^h{}_{j0} = 0$ will be called the canonical metrical $N-$linear connection on E. It will be denoted by $\overset{c}{D}\Gamma$.

The $N-$linear connection $\overset{c}{D}\Gamma$ is completely determined by the time-dependent Lagrangian L. Thus $\overset{c}{D}\Gamma$ is similar to the connection $C\Gamma$ in Lagrange spaces [3].

The $h-$ and $v-$covariant derivatives of $C = y^i \dot{\partial}_i$, with respect to $\overset{c}{D}\Gamma$ lead us to introduce the following deflection tensors for D:

$$D^i{}_o = y^i_{|0}, \quad D^i{}_k = y^i_{|k}, \quad d^i{}_k = y^i|_k. \tag{6.5}$$

Setting $D_{k0} = g_{ki} y^i_{|0}$, $D_{kj} = g_{ki} y^i_{|j}$, $d_{kj} = g_{ki} d^i{}_j$, and keeping in mind that

$\overset{c}{D}\Gamma$ is metrical, one gets

$$
(6.6)\qquad
\begin{aligned}
&D_{i0|k} - D_{ik|0} = R_{ji0k}y^j - d_{ih}R^h{}_{0k},\\
&D_{hi|k} - D_{hk|i} = R_{jhik}y^j - d_{hs}R^s{}_{ik},\\
&D_{h0}|_k - d_{hk|0} = P_{jh0k} - d_{kj}P^j{}_{0k},\\
&D_{hi}|_k - d_{hk|i} = P_{jhik}y^j - D_{hj}C^j{}_{ik} - d_{hj}P^j{}_{ik},\\
&d_{ik}|_h - d_{ih}|_k = S_{jikh}y^j.
\end{aligned}
$$

We may also introduce the $h-$ and $v-$electromagnetic tensor fields, respectively,

$$
(6.7)\qquad F_{ij} = \frac{1}{2}(D_{ij} - D_{ji}), \quad f_{ij} = \frac{1}{2}(d_{ij} - d_{ji}).
$$

As it is easy to check, $f_{ij} = 0$. As for F_{ij}, a direct calculation gives the following result.

Theorem 6.3. *The tensor field F_{ij}, given by* (6.7) *satisfies the following Maxwell equations*

$$
(6.8)\qquad
\begin{aligned}
&F_{ij|k} + F_{jk|i} + F_{ki|j} = -\sum_{(i,j,k)} R^h{}_{jk}C_{ish}y^s,\\
&F_{ij}|_k + F_{jk}|_i + F_{ki}|_j = 0,
\end{aligned}
$$

7 Some time-dependent lagrangians

Let $\widetilde{TM}$ be the manifold of nonvanishing vectors on M and let $F : \mathbb{R}\times TM \to \mathbb{R}$ be a smooth function on $\mathbb{R}\times\widetilde{TM}$ and only continuous at the points $(t, x, 0)$. Assume F is positive on $\mathbb{R}\times\widetilde{TM}$ and homogeneous of degree one with respect to y.

A quadratic form is defined by

$$
(7.1)\qquad h_{ij}(t, x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2.
$$

If this is positive definite, h_{ij} will be called a rheonomic Finsler metric on M and the pair $RF^n = (M, F)$ will be called a rheonomic Finsler space. If we set $L = F^2$, it turns out that (M, L) is a rheonomic Lagrange space. Thus, we may study the geometry of RF^n regarding it as a rheonomic Lagrange space whose Lagrangian L is positive, differentiable only on $\mathbb{R} \times \widetilde{TM}$ and homogeneous of degree two with respect to y.

Thus, the canonical nonlinear connection for (M, L) will be called the Cartan nonlinear connection of RF^n, and the canonical metrical $N-$linear connection of (M, L) will be called the Cartan metrical connection of RF^n, in such a way that the terminology corresponds to that from Finsler geometry [12]. By the Euler theorem on homogeneous functions, one finds

$$L = F^2 = h_{ij}(t, x, y) y^i y^j. \tag{7.2}$$

Introducing the Cartan tensor fields of RF^n,

$$C_{ijk} = \frac{1}{2} \dot{\partial}_i h_{jk}, \quad C_{ij0} = \frac{1}{2} \frac{\partial g_{ij}}{\partial t}, \quad C_{0ijk} = \partial_0 C_{ijk}, \tag{7.3}$$

where ∂_0 do stands for $\dfrac{\partial}{\partial t}$ the same theorem leads to the next proposition.

Proposition 7.1. *The following identities hold:*

$$\begin{aligned} &y^i C_{ijk} = y^i C_{jik} = y^i C_{jki} = 0, \\ &y^i C_{0ijk} = y^i C_{0jik} = y^i C_{0jki} = 0. \end{aligned} \tag{7.4}$$

Introducing the usual Christoffel symbols,

$$\gamma^i{}_{jk} = \frac{1}{2} h^{ir} (\partial_j h_{rk} + \partial_k h_{jr} - \partial_r h_{jk}),$$

we may state the following result.

Theorem 7.1. *The local coefficients of the Cartan nonlinear connection are as follows*

$$N^i_0 = \frac{1}{2} h^{ik} \partial_0 \dot{\partial}_k F^2, \quad N^i{}_k = \dot{\partial}_k G^i, \text{ where} \tag{7.5}$$

$$G^i = \frac{1}{2} \gamma^i{}_{jk} y^j y^k. \tag{7.6}$$

Theorem 7.2. *The Cartan metrical connection* $\overset{c}{F}T = (\overset{c}{F}{}^i{}_{j0}, \overset{c}{F}{}^i{}_{jk}, \overset{c}{C}{}^i{}_{jk})$ *is as follows:*

$$\begin{aligned} &\overset{c}{F}{}^i{}_{j0} = \frac{1}{2} h^{ik} \delta_0 h_{kj}, \\ &\overset{c}{F}{}^i{}_{jk} = \frac{1}{2} h^{is} (\delta_j h_{sk} + \delta_k h_{js} - \delta_s h_{jk}), \\ &\overset{c}{C}{}^i{}_{jk} = h^{is} C_{sjk}, \end{aligned} \tag{7.7}$$

where δ_j is constructed by the help of (7.5).

The proofs are achieved by direct calculation. Also, by a direct calculation, one gets

$$
(7.8)\qquad \begin{aligned} &N_0^i y^i = \partial_0 F^2, \quad \text{with } y_i = h_{is} y^s,\\ &y^i{}_{|k} = 0, \quad F^2{}_{|k} = 0, \quad y^i|_k = \delta^i{}_k, \quad F^2|_k = 2h_{ik} y^i. \end{aligned}
$$

By (7.8), the deflection tensors of an RF^n space are $D_{ij} = 0$, $d_{ij} = h_{ij}$, and thus the $h-$ and $v-$electromagnetic tensors identically vanishes. Thus, no rheonomic Finsler space supports a theory of electromagnetism.

It is clear that h_{ij} from (7.1) is $0-$homogeneous with respect to y. This fact suggests that we consider rheonomic Lagrange spaces whose metrical tensor fields are $0-$homogeneous with respect to y, that is, the functions g_{ij} given by (3.1) are $0-$homogeneous with respect to y. As to the general form of the Lagrangians of these spaces, we have the following theorem.

Theorem 7.3. *If the metrical tensor field (g_{ij}) of a space $RL^n = (M, L)$ is $0-$homogeneous with respect to y, then L has the general form*

$$
(7.9)\qquad L(t, x, y) = g_{ij}(t, x, y) y^i y^j + A_i(t, x) y^i + U(t, x),
$$

where A_i is a covector field and U is a real function on $\mathbb{R} \times M$.

Proof. If we put $\overset{\circ}{L}(t, x, y) = g_{ij}(t, x, y) y^i y^j$, by the homogeneity of g_{ij}, it follows $\dot{\partial}_i \dot{\partial}_j (L \overset{\circ}{L}) = 0$, which implies (7.9). ■

The following time-dependent Lagrangian, a particular form of (7.9),

$$
(7.10)\qquad L(t, x, y) = a_{ij}(t, x) y^i y^j + A_i(t, x) y^i + U(t, x),
$$

where (a_{ij}) is a time-dependent Riemann metric, was used in treating some problems of dynamics [15-17].

Let's apply our previous theory to a rheonomic Lagrange space with the time-dependent Lagrangian (7.10). First, we note that its fundamental metric tensor field is just $(a_{ij}(t, x))$. The canonical nonlinear connection is given by

$$
(7.11)\qquad \begin{aligned} &N_0^k(t, x, y) = a^{kh}(\partial_0 a_{hi}) y^i + \partial_0 a(t, x),\\ &N_h^k(t, x, y) = a^k{}_{hi}(t, x) y^i - a^{kj} A_{hj}, \end{aligned}
$$

where

$$
(7.12)\qquad A_{jh} = \frac{1}{2}\left(\frac{\partial A_j}{\partial x^h} - \frac{\partial A_h}{\partial x^j}\right),
$$

and $a^k{}_{hi}$ denotes the Christoffel symbols constructed with $(a_{ij}(t, x))$.

The canonical metrical $N-$linear connection is $\overset{c}{D}\Gamma = (0, a^i{}_{jk}(t, x), 0)$. The covariant deflection tensor fields are as follows: $D_{ij} = A_{ij}$, $d_{ij} = a_{ij}$. It comes out that $F_{ij} = A_{ij}$ and the term of $h-$electromagnetic tensor for F_{ij} is

supported by the form of A_{ij}. The Maxwell equations reduce to the classical ones.

Finally, we mention the possibility of ignoring that the metric tensor $g_{ij}(t,x,y)$ of a rheonomic Lagrange space is provided by a regular time-dependent Lagrangian, and study the geometry of the pair $(M, g_{ij}(t,x,y))$. Many results discussed in the above may be extended to this more general setting (cf. [5, Ch. XIII]).

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CERTAIN GENERALIZATIONS OF FINSLER METRICS

by Mihai ANASTASIEI

1 Introduction

Scrutinizing the main body of results from Finsler geometry it is observed that many of them depend on the Finsler metric only and not on the fundamental Finsler function. Moreover, there are many such results in which only some basic properties of the Finsler metric are involved.

These facts led R. Miron ([9] [11]) to propose the study of generalized Lagrange metrics, GL–metrics for brevity, whose definition is tailored after the basic properties of Finsler metrics.

The geometry of these metrics proved to be useful in the Theory of General Relativity, Gauge Theory, and Ecology (cf. [11] and references therein).

Certain problems from Mechanics and Theoretical Physics require one, even at the Finslerian level, to study the geometry of a GL–metric which, furthermore, depends on a special variable analogous to the physical time. We contributed to this study in [4].

The main objective of this paper is to review from our own viewpoint the generalizations of Finsler metrics mentioned above. We take this opportunity to cast a new light on some well–known results and to add several new ones. Some new examples are provided, too.

2 Some properties of the Finsler metric

Let M be a real, smooth i.e. C^∞, finite dimensional manifold and $\tau : TM \to M$ its tangent bundle. Set $\overset{\circ}{T} M = TM \setminus \{0_x \in T_xM,\, x \in M\}$. Let $(U, (x^i))$ be a local chart on M. The indices $i, j, k, ...$ will run from 1 to $n = \dim M$ and the Einstein convention on summation will be implied. Associate to $v \in \tau^{-1}(U)$ the coordinates $(x^i(\tau(u))$ and (y^i) provided by $v_{\tau(v)} = y^i\partial_i$, $\partial_i := \dfrac{\partial}{\partial x^i}$ and TM becomes a smooth orientable manifold. A change of coordinates $(x^i, y^i) \to (x^{i'}, y^{i'})$ on TM is as follows:

$$x^{i'} = x^{i'}(x^1, ..., x^n),\ y^{i'} = (\partial_j x^{i'})y^j,\ \operatorname{rank}(\partial_j x^{i'}) = n. \tag{2.1}$$

Let $F^n = (M, F)$ be a Finsler space and $\gamma_{ij}(x, y)$ the local components of its Finsler metric. We list the following known properties of the Finsler metric, some of which are stated just as the definition.

P$_1$. A change of coordinates (2.1) implies

$$\gamma_{ij}(x,y) = (\partial_i x^{i'})/(\partial_j x^{j'})\gamma_{i'j'}(x',y'). \tag{2.2}$$

Thus $(\gamma_{ij}(x,y))$ are the components of a special, *distinguished* tensor field on $\overset{\circ}{T} M$ in the sense that their transformation law (2.2) is similar with that of the components of a tensor field on M. Throughout Finsler geometry and its generalizations one meets such geometrical objects i.e. defined on $\overset{\circ}{T} M$ or TM but transforming under (2.1), as being on M. We called them d–geometrical objects, [11].

P$_2$. $\gamma_{ij}(x,y) = \gamma_{ji}(x,y)$ (symmetry).

P$_3$. $\det(\gamma_{ij}(x,y)) \neq 0$ (non–degeneracy).

This property is usually postulated in a stronger form: the quadratic form $\gamma_{ij}\xi^i\xi^j$, $(\xi^i) \in \mathbb{R}^n$ is positive definite.

P$_4$. $\gamma_{ij}(x,y) = \dfrac{1}{2} \overset{\circ}{\partial}_i \overset{\circ}{\partial}_j F^2$, $\overset{\circ}{\partial}_i := \dfrac{\partial}{\partial y^i}$.

P$_5$. $\gamma_{ij}(x,y)$ are p (positively)–homogeneous functions of zero degree with respect to (y^i). Recall that F is p–homogeneous of degree 1 in y^i.

P$_6$. The function $\overset{\circ}{C}_{ijk} = \dfrac{1}{2} \overset{\circ}{\partial}_k \gamma_{ij}$ are the components of a totally symmetric d–tensor field on $\overset{\circ}{T} M$. Moreover, $y^k \overset{\circ}{C}_{ijk} = 0$.

P$_7$. Let $\overset{\circ}{\gamma}{}^i_{jk}(x,y) = \dfrac{1}{2}\gamma^{ih}(\partial_j\gamma_{hk} + \partial_k\gamma_{jh} - \partial_h\gamma_{jk})$ be the Christofell symbols derived from (γ_{jk}). Then $\overset{\circ}{G}{}^i = \dfrac{1}{2} \overset{\circ}{\gamma}{}^i_{jk} y^j y^k$ are the components of the (geodesic) spray $\overset{\circ}{S} = y^i\partial_i + \overset{\circ}{G}{}^i \overset{\circ}{\partial}_i$ on $\overset{\circ}{T} M$ and $\overset{\circ}{N}{}^i_j = \overset{\circ}{\partial}_j \overset{\circ}{G}{}^i$ has the following law of transformation under (2.1):

$$\overset{\circ}{N}{}^{i'}_{j'}(\partial_i x^{j'}) = (\partial_j x^{i'}) \overset{\circ}{N}{}^j_i - (\partial_i\partial_j x^{i'})y^j, \tag{2.3}$$

that is, these functions are the coefficients of the nonlinear Cartan connection.

Set $\overset{\circ}{\delta}_i := \partial_i - \overset{\circ}{N}{}^k_i \overset{\circ}{\partial}_k$ and it results that $\delta_i = (\partial_i x^{i'})\delta_{i'}$. For $v \in \overset{\circ}{T} M$, the linear space H_v spanned by $(\delta_i)_v$ is supplementary to the vertical space $V_v = \mathrm{Ker}(D\tau)_v$ spanned by $(\overset{\circ}{\partial}_i)_v$, that is,

$$T_v \overset{\circ}{T} M = H_v \oplus V_v \quad \text{(direct sum)}. \tag{2.4}$$

P$_8$. The function $(\overset{\circ}{L}{}^i_{jk}, \overset{\circ}{C}{}^i_{jk})$ given by

$$\begin{aligned} \overset{\circ}{L}{}^i_{jk} &= \tfrac{1}{2}\gamma^{ih}(\overset{\circ}{\delta}_j \gamma_{hk} + \overset{\circ}{\delta}_k \gamma_{jh} - \overset{\circ}{\delta}_h \gamma_{jk}), \\ \overset{\circ}{C}{}^i_{jk} &= \tfrac{1}{2}\gamma^{ih}(\overset{\circ}{\partial}_j \gamma_{hk} + \overset{\circ}{\partial}_k \gamma_{jh} - \overset{\circ}{\partial}_h \gamma_{jk}) = \gamma^{ih} \overset{\circ}{C}_{hjk}, \end{aligned} \tag{2.5}$$

are the local coefficients of the Cartan connection. This connection is h–metrical ($\gamma_{ij\overset{\circ}{|}k} = 0$), v–metrical ($\gamma^i_{jh}\overset{\circ}{\Big|}_k = 0$), h–symmetric ($\overset{\circ}{L}{}^i_{jk} = \overset{\circ}{L}{}^i_{kj}$), v–symmetric ($\overset{\circ}{C}{}^i_{jk} = \overset{\circ}{C}{}^i_{kj}$) and is free of deflection ($\overset{\circ}{D}{}^i_j = y^i_{\overset{\circ}{|}j} = 0$). In other words, it satisfies the well–known Matsumoto's axioms. The list could be continued but these properties are essential for developing Finsler geometry.

3 A generalization of the Finsler metrics: GL–metrics

A collection of functions $(g_{ij}(x,y))$ locally defined on TM and satisfying P_1–P_3 is called a generalized Lagrange metric, shortly a GL–metric. As P_7 cannot be recovered from P_1–P_3 only, we introduce the assumption

(H_1) There exists a non–linear connection on TM i.e. a set of coefficients $(N^i_j(x,y))$ verifying (2.3).

This is always true if M is paracompact. Notice that we shall indicate the general case by deleting the superscript "$\circ$" from the entities previously introduced. Thus we may consider (∂_i) and the decomposition (2.4) holds for $v \in TM$. The functions provided by (2.5) define a connection with the first four properties of the Cartan connection. Its deflection generally does not vanish. From now on the torsions, the curvatures, the h– and v–paths and so on can be introduced and studied.

The postulate (H_1) is not very strong as the hypothesis of paracompactness of M is generally accepted. But an arbitrary non–linear connection i.e. without any relationship to $(g_{ij}(x,y))$ is far from useful.

Fortunately, in the most important examples there exists a non–linear connection determined by or strongly related to the given GL–metric.

Example. For any positive functions a and b on $\overset{\circ}{T} M$ we set

$$g_{ij}(x,y) = a(x,y)\gamma_{ij}(x,y) + b(x,y)y_iy_j,\ y_i = \gamma_{ik}y^k. \tag{3.1}$$

This is a GL–metric. Indeed, it is easy to check that

$$g^{jk} = \frac{1}{a}\left(\gamma^{jk} - \frac{b}{a+bF^2}\ y^jy^k\right), \tag{3.2}$$

verifies $g_{ij}g^{jk} = \delta^k_i$.

In order to study it we have on hand the non–linear connection $(\overset{\circ}{N}{}^i_j(x,y))$. We stress that for various functions a and b the GL–metric (3.1) supplies all the GL-metrics treated in [11].

A GL-metric is said to be an L–metric if there exists a smooth function $L : TM \to R$ such that

$$g_{ij}(x,y) = \frac{1}{2}\ \overset{\circ}{\partial}_i\overset{\circ}{\partial}_j\ L(x,y). \tag{3.3}$$

Such a function, called a regular Lagrangian, exists if and only if (C_{ijk}) is a totally symmetric d–tensor field. If L exists, it is not unique since $\tilde{L}(x,y) = L(x,y) + \varphi_i(x)y^i + c$ is a new solution of (3.3). Choosing such an L, the pair (M, L) is called a Lagrange space. In particular, (M, F^2) is a Lagrange space.

For L–metrics, a canonical non–linear connection is derived from the Euler-Lagrange equations provided by the variational problem $\delta \int_{t_0}^{t_1} L dt = 0$, by first considering $G^i = \frac{1}{4} g^{ik}[(\overset{\circ}{\partial}_k \ \partial_j L)y^j - \partial_k L]$ (the components of the canonical semi–spray) and then taking $N^i_j = \overset{\circ}{\partial}_j G^i$.

If one requires that a L–metric be $(m-2)-p-$homogeneous, then L is uniquely determined and is $m-(p)-$homogeneous. For such L–metrics the functions G^i and the connection (L^i_{jk}, C^i_{jk}) is deflection free. Thus these L–metrics are closely related to Finsler metrics, [5], [6].

Coming–back to the Example, we notice that $(g_{ij}(x,y))$ is an L–metric if and only if

$$[(\overset{\circ}{\partial}_k a)\gamma_{ij} - (\overset{\circ}{\partial}_i a)\gamma_{kj}] + [(\overset{\circ}{\partial}_k b)y_i - (\overset{\circ}{\partial}_i a)y_k]y_j + b[y_i\gamma_{kj} - y_k\gamma_{ij}] = 0. \tag{3.4}$$

Contracting this by (γ^{ij}) one gets

$$(n-1)(\overset{\circ}{\partial}_k a) - b(n-1)y_k + (\overset{\circ}{\partial}_k b)F^2 - (\overset{\circ}{\partial}_i b)y^i y_k = 0. \tag{3.5}$$

Remark 3.1. Even for simple functions a and b, the GL-metric (3.1) does not reduce to an L–metric. For instance, if a and b are positive constants, (3.5) simplifies to $b(n-1)y_k = 0$, which does not hold for $n > 1$. So (3.4) fails.

Remark 3.2. Let $a = \alpha(F^2)$ and $b = \beta(F^2)$ with $\alpha, \beta : I\mathbb{R}^*_+ \to \mathbb{R}^*_+$. Then (3.5) implies $\beta = 2\alpha'$ and the condition $a + bF^2 > 0$ becomes $\alpha + 2\alpha' t > 0$, $t \to \alpha(t)$, $t \in \mathbb{R}^*_+$. Set $\alpha = \varphi'$ with $\varphi : \mathbb{R} \to \mathbb{R}^*_+$, $\varphi' > 0$, $\varphi'(t) + 2\varphi'' t > 0$. One obtains the φ–Lagrange metrics studied in [6].

4 Almost Hermitian Model of a GL-metric

Let M be endowed with a GL-metric $(g_{ij}(x,y))$ and a non–linear connection $(N^i_j(x,y))$. The decomposition (2.4) implies a decomposition $X = hX + vY$ for every vector field (v.f.) X on TM. Denote by P the almost product structure provided by the horizontal and vertical distributions according to: $P(hX) = hX$, $P(vX) = -vX$. Consider also the almost complex structure F defined as follows: $F(hX) = -vX$, $F(vX) = hX$. Next, setting $G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$, $\delta y^i = dy^i + N^i_k dx^k$, one gets a metrical structure on TM which is a Riemannian structure if (g_{ij}) is positive definite. It is easily seen that (F, G) is an almost Hermitian structure.

This simple construction has much more implications in the geometry of $(g_{ij}(x,y))$ then it seems at a first insight. Indeed, the coefficients (L^i_{jk}, C^i_{jk}) supply a linear connection D on TM given in the adapted basis $(\delta_i, \overset{\circ}{\partial}_i)$ as follows:

$$\begin{aligned} D_{\delta_k}\delta_j &= L^i_{jk}\delta_i, & D_{\overset{\circ}{\partial}_k}\delta_j &= C^i_{jk}\delta_i, \\ D_{\delta_k}\overset{\circ}{\partial}_j &= L^i_{jk}\overset{\circ}{\partial}_i, & D_{\overset{\circ}{\partial}_k}\overset{\circ}{\partial}_j &= C^i_{jk}\overset{\circ}{\partial}_i\ . \end{aligned} \tag{4.1}$$

This connection preserves the both distributions ($DP = 0$), is almost complex ($DF = 0$) and is metrical ($DG = 0$). Moreover, if its torsion T is decomposed into vertical and horizontal components, then $hT(h\cdot, h\cdot) = 0$ and $vT(v\cdot, v\cdot) = 0$. Conversely, every linear connection D on TM, with the above properties, has (L^i_{jk}, C^i_{jk}) from (2.5) as local coefficients in the adapted basis $(\delta_i, \overset{\circ}{\partial}_j)$. Thus the study of the connection (2.5) is equivalent to the study of such a linear connection D on TM endowed with (F, G). This is a reason to call (F, G) the almost Hermitian model of $(g_{ij}(x,y))$. A first important application of this model is due to R. Miron. He considered the Einstein equations for G and projecting them on horizontal and vertical distributions, he arrived at a correct form of the Einstein equations for $(g_{ij}(x,y))$, [9]. See also [2],[3],[11],[12].

A simpler usage of the almost Hermitian model is as follows. Looking for a meaning of the divergence of a d–vector field $(X^i(x,y))$ we observe that it defines an horizontal vector field $hX = X^i(x,y)\delta_i$ as well as a vertical vector field $vX = X^i(x,y)\,\overset{\circ}{\partial}_i$. As (TM, G) is an orientable Riemannian manifold (the positiveness of (g_{ij}) is implied), we define an h–divergence $(\mathrm{div}_h X)$ and a v–divergence $(\mathrm{div}_v X)$ according to $\mathcal{L}_{*X}dv = (\mathrm{div}_* X)dv$, $* = h, v$, where $\mathcal{L}$ means the Lie derivative and dv is the volume element associated to G.

Since D has torsion, it comes out that the usual formula for the divergence of a vector field Z on TM is $\mathrm{div}\, Z = Trace(Y \to D_Y Z + T(Z, Y))$. In the adapted basis one finds $\mathrm{div}_h X = X^i_{|i} - X^k P_k$, $P_k = P^i_{ki}$, $P^i_{jk} = \overset{\circ}{\partial}_k N^i_j - L^i_{kj}$, $\mathrm{div}_v X = X^i\big|_i + X^k C_k$, $C_k = C^i_{ki}$. For the L–metrics described in Remark 3.2, in particular, for Finsler metrics we get $\mathrm{div}_h S = 0$, a generalization of a Liouville theorem from the Riemannian geometry. For any function f on TM we have a h–gradient $\mathrm{grad}_h f = (g^{ik}\delta_k f)\delta_i$ and a v–gradient: $\mathrm{grad}_v f = (g^{ik}\,\overset{\circ}{\partial}_k f)\,\overset{\circ}{\partial}_i$. Accordingly, we may define the h–Laplacean $\Delta_h f = \mathrm{div}_h(\mathrm{grad}_h f)$ and the v–Laplacean $\Delta_v f = \mathrm{div}_v(\mathrm{grad}_v f)$.

The function $\varepsilon = g_{ij}(x,y)y^i y^j$ is called the absolute energy of the GL-metric $(g_{ij}(x,y))$. For L–metrics discussed in the Remark 3.2, the absolute energy is h–harmonic i.e. $\Delta_h \varepsilon = 0$.

Let $\mathbb{C} = y^i\,\overset{\circ}{\partial}_i$ be the Liouville vector field on TM. The postulate (H_1) is clearly *implied* by the following one.

(H_2) There exists a linear connection ∇ in the vertical bundle which is regular, that is, the space $H_v = \{X_v \mid \nabla_{X_v}\mathbb{C} = 0\}$ is supplementary to V_v, $v \in TM$.

Let h_v be the inverse of the isomorphism $D_v\tau : H_v \to T_{\tau(v)}M$ and $\delta_i = h_v(\partial_i)$. Then $(D_v\tau)(\delta_i - \partial_i) = 0$. Thus $\delta_i = \partial_i - N_i^k \overset{\circ}{\partial}_k$, where the sign "$-$" is taken for the sake of convenience. The functions (N_i^k) define a nonlinear connection.

Let the linear connection ∇ be given as follows: $\nabla_{\partial_k} \overset{\circ}{\partial}_j = \Gamma^i_{jk} \overset{\circ}{\partial}_i$, $D_{\overset{\circ}{\partial}_k} \overset{\circ}{\partial}_j = B^i_{jk} \overset{\circ}{\partial}_i$. The condition $\nabla_{\delta_k}(y^j \overset{\circ}{\partial}_j) = 0 \iff (\delta^i_h + y^j A^i_{jh})N^h_k = y^j\Gamma^i_{jk}$ shows that the regularity of ∇ is equivalent to the regularity of the matrix $(\delta^i_h + y^j A^i_{jh})$. The triad $(N^i_j, L^i_{jk} = \Gamma^i_{jk} - N^h_k B^i_{jh}, B^i_{jk})$ is an usual Finsler connection.

The postulate (H_2) is involved in what we called the vector bundle model of Finsler geometry (cf. [1]). This model was recently used by D.Bao, S.S.Chern [7] and Z.Shen [13] for solving some global problems in Finsler geometry. A variant of it, usefull for Physics, was developed by J.G. Vargas and D.Torr [14]. An essentially different model, called by us the principal bundle model (cf. [1]), is due to M. Matsumoto [8].

5 Finsler geometry of a vector bundle

It is to be observed that the geometry of a GL–metric essentially depends on a non–linear connection on TM. The extension of this notion of connection to a vector bundle $\pi : E \to M$ is quite natural. It is nothing but a supplementary distribution to the vertical distribution $u \to (\operatorname{Ker} D\pi)_u$, $u \in E$. Notice that the horizontal distribution is non–holonomic, so a study of this pair of distributions is of interest.

If E is endowed with a metrical structure G, we may take as non–linear connection the orthogonal distribution to the vertical distribution. Then G takes the form

$$G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ab}(x,y)\delta y^a \otimes \delta y^b,\ \ \delta y^a = dy^a + N^a_k dx^k, \tag{5.1}$$

where (x^i, y^a), $a, b, c, ... = 1, ..., m$ = fibre dimension, are the local coordinates on E and $(N^a_i(x,y))$ are the local coefficients of the non–linear connection defined by G.

A change of coordinates $(x^i, y^a) \to (x^{i'}, y^{a'})$ on E has the form

$$\begin{aligned} &x^{i'} = x^{i'}(x^1, ..., x^n), && \operatorname{rank}(\partial_j x^{x'}) = n,\\ &y^{a'} = M^{a'}_b(x)y^b, && \operatorname{rank}(M^{a'}_b) = m. \end{aligned} \tag{5.2}$$

The coefficients (N^a_i) of a non–linear connection have the following transformation law under (5.2):

$$N^{a'}_{i'}(\partial_i x^{i'}) = M^{a'}_a(x)N^a_i - (\partial_i M^{a'}_a(x))y^a. \tag{5.3}$$

A geometrical study of the pair (E, G) using the above ingredients was performed by R. Miron [10]. Some applications of his theory we have pointed out in [2],[3] (see also [11]).

6 Rheonomic GL–metrics

Let us consider the functions $(g_{ij}(t,x,y))$ with the properties of a GL–metric. Assume t remains unchanged under (2.1), that is, t is viewed as absolute time. We call such a collection of functions a *rheonomic GL–metric*, shortly a RGL–metric. It is clear that this kind of GL–metric is living on $\mathbb{R}\times TM$, a manifold which could be thought of as fibered in three different ways projecting it on $\mathbb{R}$, TM or $\mathbb{R}\times TM$. Each of these fibrations has a certain value for geometrizing problems from Mechanics or Calculus of Variations. As more appropriate seems to be the fibration

$$\pi:\mathbb{R}\times TM\to\mathbb{R}\times M,\ \pi(t,v)=(t,\tau(v)),\ v\in TM.$$

Set $E=\mathbb{R}\times TM$. The manifold E is coordinizated by (t,x^i,y^i) and the π takes the form $(t,x^i,y^i)\to(t,x^i)$. It is convenient to put $x^0=t$ and to use the Greek indices $\alpha,\beta,\gamma,...$ ranging over $0,1,2,...,n$. A non–linear connection can be given by $(n+1)$ local vector fields, say δ_α. Choosing δ_α such that they are projected to ∂_α, one gets

$$\delta_\alpha=\partial_\alpha-N^i_\alpha(t,x,y)\overset{\circ}{\partial}_i\ . \tag{6.1}$$

The invariance of the horizontal subspaces requires the condition $\delta_\alpha=(\partial_\alpha x^{\alpha'})\delta_{\alpha'}$, when a change of coordinates on $\mathbb{R}\times TM$ is performed. This implies the following law of transformation for (N^i_α):

$$N^{i'}_{\alpha'}(\partial_\beta x^{\alpha'})=(\partial_k x^{i'})N^k_\beta-(\partial_\beta\partial_k x^{i'})y^k. \tag{6.2}$$

If one rewrites (6.1) in the form

$$\delta_0=\partial_t-N^i_0(t,x,y)\overset{\circ}{\partial}_i,\ \ \delta_i=\partial_i-N^k_i(t,x,y)\overset{\circ}{\partial}_k,$$

it comes out from (6.2) that $(N^i_0(t,x,y))$ change like the components of a d–vector field and $(N^i_j(t,x,y))$ change like the coefficients of a non–linear connection on TM. Thus, we may identify a non–linear connection on E with the pair (N^i_0,N^i_j).

Let $(\delta_0,\delta_i,\overset{\circ}{\partial}_i)$ be the basis adapted to the decomposition $T_uE=H_uE\oplus V_uE$ and $(dt,dx^i,\delta y^i)$ its dual. Denote by P the almost product structure on E associated as in §4 to the decomposition $T_uE=N_uE\oplus V_uE$ and define a tensor field Φ of type $(1,2)$ on E as follows:

$$\Phi(\delta_0)=0,\ \Phi(\delta_i)=-\overset{\circ}{\partial}_i,\ \Phi(\overset{\circ}{\partial}_i)=\delta_i. \tag{6.3}$$

It easily comes out that $(\Phi,\delta_0,\delta t)$ is an almost contact structure on E. Using $(g_{ij}(t,x,y))$ the following metrical structure on G is obtained

$$G=dt\otimes dt+g_{ij}dx^i\otimes dx^j+g_{ij}\delta y^i\otimes\delta y^j. \tag{6.4}$$

It is easy to see that (Φ,δ_0,dt,G) is a metrical almost contact structure on E. This will be called the almost contact model for $g_{ij}(t,x,y)$. As in the

almost Hermitian model it is quite natural to look for a linear connection D on E with the properties:

$$\text{(6.5)}\quad DP=0,\ D\Phi=0,\ DG=0,\ D\delta_0=0,\ hT(h\cdot,h\cdot)=0,\ vT(v\cdot,v\cdot)=0.$$

In the frame $(\delta_0,\delta_i,\overset{\circ}{\partial}_i)$ this connection has the coefficients $(L^i_{j0},L^i_{jk},C^i_{jk})$, where the latter two are similar with those from (2.5) while the first has the form

$$L^i_{j0}=\frac{1}{2}\,g^{ih}\delta_0 g_{hj}+\frac{1}{2}(\delta^k_j\delta^i_h-g_{jh}g^{ki})X^h_{k0}, \tag{6.6}$$

with X^h_{k0} an arbitrary d–tensor field, cf. [4].

This set of connections allows us to develop the geometry of the RGL–metric $g_{ij}(t,x,y)$.

As for GL–metrics, in the most important examples, the non–linear connection is completely determined by (g_{ij}). A RGL–metric will be called a rheonomic L–metric if there exists a smooth function $L:\mathbb{R}\times TM\to\mathbb{R}$ such that

$$g_{ij}(t,x,y)=\frac{1}{2}\ \overset{\circ}{\partial}_i\overset{\circ}{\partial}_j\ L. \tag{6.7}$$

If L exists, it is not unique. Taking one L as solution of (6.6), the pair (M,L) is called a rheonomic Lagrange space. In particular, we arrive at the notion of theonomic Finsler space as a pair (M,F) with $F:R\times TM\to R$ a positive function, smooth on $R\times\overset{\circ}{T}M$, p–homogeneous of degree 1 with respect to (y^i) such that the functions $g_{ij}(t,x,y)=\frac{1}{2}\overset{\circ}{\partial}_i\overset{\circ}{\partial}_j F^2$ satisfy $\det(g_{ij}(t,x,y))\neq 0$. For a theory of rheonomic Finsler and Lagrange spaces we refer to [4].

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Faculty of Mathematics
"Al.I.Cuza" University of Iaşi
R–6600 Iaşi, Romania
E–mail: anastas@uaic.ro

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A HISTORICAL REMARK ON THE CONNECTIONS OF CHERN AND RUND

BY

M. ANASTASIEI

1 Introduction

Let M be a real, $n-$dimensional smooth (i.e. C^∞) manifold and $\tau : TM \to M$ its tangent bundle. In a local chart (U, x^i) on M, a tangent vector $v \in T_pM$, $p \in M$, has the form $v = y^i \left.\dfrac{\partial}{\partial x^i}\right|_p$ and it is usual to take $(\tau^{-1}U, x^i \equiv x^i \circ \tau, y^i)$ as local coordinates on TM. Throughout the paper the indices run from 1 to n and the Einstein convention on summation is implied.

A local change of coordinates $x^i \to \widetilde{x}^i$ on M induces in turn a change of coordinates $(x^i, y^i) \to (\widetilde{x}^i, \widetilde{y}^i)$ on TM:

$$
\begin{aligned}
&\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \ \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^k}\right) = n,\\
&\widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^k}(x) y^k.
\end{aligned} \tag{1.1}
$$

Set $(x, y) := (x^i, y^i)$ and $\overset{\circ}{TM} = TM \setminus \{(x, 0)\}$.

A fundamental Finsler function is a function $F : TM \to \mathbb{R}$, $(x, y) \to F(x, y)$, with the properties
(1.2) $F(x, y) \geq 0$ with equality if and only if $y = 0$,
(1.3) F is smooth on $\overset{\circ}{TM}$ and only continuous on $TM \setminus \overset{\circ}{TM}$,
(1.4) $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$,
(1.5) $g_{ij}(x, y)\xi^i\xi^j \geq 0$ with equality if, and only if, $(\xi^i) = 0$, where

$$
g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{1.6}
$$

The pair $F^n = (M, F)$ is called a Finsler space. Its geometry is called Finsler geometry.

The geometrical objects from Finsler geometry are in fact living on the sphere bundle $SM \to M$, $SM = TM/\sim$, $(x, y) \sim (x, \widetilde{y})$ if, and only if, there

exists an $a > 0$ such that $\widetilde{y} = ay$. However, for convenience we shall work with the slit tangent bundle $\overset{\circ}{TM}$ instead of SM.

The equivalence problem in Finsler geometry is to decide whether two fundamental Finsler functions F and $\widetilde{F}$ will transform into each other under a diffeomorphism $(x, y) \to (\widetilde{x}, \widetilde{y})$. In order to solve this problem using E. Cartan's equivalence method, S.S. Chern has introduced in 1948, [1], a remarkable connection in Finsler geometry by means of some connection $1-$forms. That connection remained outside of the mainstream of the development of Finsler geometry in the next decades. It was only briefly treated in the monograph by H. Rund, [6], and not at all in those of M. Matsumoto [4], R. Miron and M. Anastasiei [5]. Chern came back to his connection in 1992, [2]. Then, in a large joint paper with Bao, [3], its extraordinary usefulness in treating global problems in Finsler geometry was shown.

This fact appeared quite strange to us since along years of study of Finsler geometry and its generalizations we thought of and experienced a mechanism of producing Finsler connections. Thus we decided to see what is the place of Chern's connection among all Finsler connections.

Let $\tau^*TM \to \overset{\circ}{TM}$ be the pull-back of the tangent bundle by τ. An interpretation of Chern's connection as a linear connection in this pull-back bundle has been sketched in [?]. We have, however, chosen to relate it to the Cartan nonlinear connection associated to F. This allows us to view Chern's connection as a Finsler connection, [2], or in the terminology from [5] as a normal $d-$connection. Quite surprisingly we arrived at the Rund connection as defined in [?], [2], [5]. Thus in the famous diagram involving the four remarkable Finsler connection: Berwald, Rund, Cartan, Hashiguchi, [2], p. 120, Rund's name has to be replaced by Chern's who discovered the connection in question almost ten years earlier. In fact, Rund had a little bad luck with this connection (cf. Remark 18.1 in [2]). These facts do not diminish at all the contribution of Rund and any history of Finsler geometry has to put his name on an outstanding place.

The structure of the paper is as follows. In §2, we recall Chern's connection $1-$forms. Then in §3, viewing Chern's connection as a Finsler connection we show that it coincides with the Rund connection.

2 The Chern connection $1-$form

We follow [?] for recalling the definition and some properties of Chern's connection $1-$forms.

Set $\partial_i := \dfrac{\partial}{\partial x^i}$, $\overset{\circ}{\partial}_i := \dfrac{\partial}{\partial y^i}$.

The homogeneity stipulation (1.4) implies

$$y^i \overset{\circ}{\partial}_i F = F, \tag{2.1}$$

$$y^i \overset{\circ}{\partial}_i \overset{\circ}{\partial}_i F = 0, \tag{2.2}$$

$$y^i g_{ij} = \frac{1}{2} \overset{\circ}{\partial}_j F^2, \tag{2.3}$$

$$F^2 = g_{ij} y^i y^j, \tag{2.4}$$

$$y^i C_{ijk} = 0,\ C_{ijk} = \frac{1}{2} \overset{\circ}{\partial}_k \overset{\circ}{\partial}_i \overset{\circ}{\partial}_j F^2. \tag{2.5}$$

By (1.1) and (1.5) it follows that, as a well-defined $(0,2)-$tensor field on $\overset{\circ}{TM}$,

$$g = g_{ij} dx^i \otimes dx^j \tag{2.6}$$

is symmetric and positive definite.

The sections of $\tau^* TM \to \overset{\circ}{TM}$ will be called $\tau-$vector fields or vector fields along τ. Let $\chi(\tau)$ be the set of all $\tau-$vector fields. The fibre of $\tau^* TM \to \overset{\circ}{TM}$ at $u \in \overset{\circ}{TM}$ is $T_{\tau(u)} M$. It has a basis $\left(\frac{\partial}{\partial x^i} \right)_{\tau(u)}$ and an inner product given by (2.6). A $\tau-$vector field $\overline{X} \in \chi(\tau)$ is locally given as $\overline{X} = X^i(x,y) \left(\frac{\partial}{\partial x^i} \right)$, the components $(X^i(x,y))$ being smooth functions and transforming under (1.1) as follows

$$\widetilde{X}^i(\widetilde{x}, \widetilde{y}) = \frac{\partial \widetilde{x}^i}{\partial x^k} = \frac{\partial \widetilde{x}^i}{\partial x^k}(x) X^k(x,y). \tag{2.7}$$

This suggests that we take into consideration $\mathbb{T} = y^i \left(\frac{\partial}{\partial x^i} \right)_{\tau(u)}$ as a remarkable element of $\chi(\tau)$.

By (2.4) and (2.7) one gets

$$g(\mathbb{T}, \mathbb{T}) = F^2 \tag{2.8}$$

i.e. the length of $\mathbb{T}$ is just F.

Let $\{e_i\}$ be a local orthonormal (with respect to g) frame field for the vector bundle $\tau^* TM \to \overset{\circ}{TM}$ such that $e_n = \frac{y^i}{F} \frac{\partial}{\partial x^i}$ and $\{w^i\}$ its dual $\omega-$frame. One finds that $\omega^n = (\overset{\circ}{\partial}_i F) dx^i$. Let us set

$$\omega^i = v^i_k dx^k,\ e_j = u^h_j \partial_h \tag{2.9}$$

$$dx^i = u^i_k \omega^k,\ \partial_j = v^h_j e_h. \tag{2.10}$$

These show that ω^i and e_j can be regarded as $1-$forms and vector fields on $\overset{\circ}{TM}$, respectively.

According to [?], §2, there exists a set of 1−forms ω_j^i on $T\overset{\circ}{M}$ such that

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{2.11}$$

$$\omega_{ik} + \omega_{ki} := \omega_i^j \delta_{jk} + \omega_k^j \delta_{ji} = -H_{ikj}\omega_n^j, \tag{2.12}$$

$$H_{abc} v_i^a v_j^b v_k^c = F \overset{\circ}{\partial}_k g_{ij}. \tag{2.13}$$

The 1−forms ω_j^i define Chern's connection. We do not write down the fairly complicated expression of these 1−forms in which the partial derivatives of F are involved. We notice only the following combinations of these partial derivatives which will be used later.

$$G_i := \frac{1}{4}(y^k \overset{\circ}{\partial}_i \partial_k F^2 - \partial_i F^2), \tag{2.14}$$

$$G^i = g^{ik} G_k, \tag{2.15}$$

$$G_j^i := \overset{\circ}{\partial}_j G^i. \tag{2.16}$$

In the structure equation (2.11), d means the exterior differentiation on TM.

Let Γ_i^j be the representation of ω_i^j in the natural frame. One defines a covariant differentiation ∇ by

$$\nabla \partial_k = \Gamma_k^i \otimes \partial_i, \tag{2.17}$$

and one proves that

$$\Gamma_k^i = \Gamma_{kh}^i dx^h, \ \Gamma_{kh}^i = \Gamma_{hk}^i, \tag{2.18}$$

and with $\Gamma_{kh}^i = g^{ij}\Gamma_{jkh}$ one finds

$$\Gamma_{jkh} = \frac{1}{2}(\partial_h g_{jk} - \partial_j g_{kh} + \partial_k g_{hj}) + \frac{1}{2}(M_{jkh} - M_{khj} + M_{hjk}), \tag{2.19}$$

where

$$M_{jkh} = -G_h^t \overset{\circ}{\partial}_t g_{jk}. \tag{2.20}$$

3 Chern's connection as Finsler connection

Recall that according to [4], Ch. I, III, a Finsler connection (a normal $d-$connection in [5], Ch. VII) is a triad $(N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ where $(N^i_j(x,y))$ are the local coefficients of a nonlinear connection, $F^i_{jk}(x,y)$ behave like the coefficients of a linear connection and $C^i_{jk}(x,y)$ are the components of a tensor field. Such a connection is called $h-$metrical if

$$g_{ij|k} := \delta_k g_{ij} - F^h_{ik} g_{hj} - F^h_{jk} g_{ih} = 0, \tag{3.1}$$

where $\delta_k = \partial_k - N^i_k \overset{\circ}{\partial}$, and $v-$metrical if

$$g_{ij}|_k := \overset{\circ}{\partial}_k g_{ij} - C^h_{ik} g_{hj} - C^h_{jk} g_{ih} = 0. \tag{3.2}$$

Now we shall regard Chern's connection as a Finsler connection showing that it is a $h-$metrical one. First, we re-express Γ^i_{kj} as follows. Considering $\delta_i = \partial_i - G^j_i \overset{\circ}{\partial}_j$ we observe that $\delta_i g_{kh} = \partial_i g_{kh} + M_{khi}$. Inserting this in (2.19) we find

$$\Gamma_{jkh} = \frac{1}{2}(\delta_k g_{jh} + \delta_h g_{jk} - \delta_j g_{kh}). \tag{3.3}$$

Now we must check that (Γ^i_{kh}) behave like (F^i_{kh}) under (1.1). But we can avoid this complicated calculation as we shall see below.

For the covariant differentiation of g with respect to Chern's connection we have $(\nabla g)(\partial_i, \partial_j) = dg_{ij} - \Gamma^k_i g_{kj} - \Gamma^k_j g_{ik} = dg_{ij} - (\Gamma^k_{ih} g_{kj} + \Gamma^k_{jh} g_{ik}) dx^h$ and using (3.1) we find $(\nabla g)(\partial_i, \partial_j) = (\overset{\circ}{\partial}_h g_{ij})(dy^h + G^h_s dx^s)$.

We put $\delta y^h = dy^h + G^h_k dx^k$ and it is easily checked that $\delta y^h(\delta_k) = 0$.

Going back to the above formulae we conclude that Chern's connection satisfies

$$dg_{ij} = \Gamma^k_i g_{kj} + \Gamma^k_j g_{ik} + 2C_{kij} \delta y^k. \tag{3.4}$$

We note also that from $(\nabla g)(\partial_i, \partial_j) = (\overset{\circ}{\partial}_h g_{ij}) \delta y^h$ it results that ∇ is metrical only for those tangent vectors v which verify $\delta y^h(v) = 0$. Recall that for $C_{ijk} = 0$, F^n reduces to a Riemannian space.

This fact motivates us to introduce the following:

Definition 3.1. A tangent vector $X_n \in T_u \overset{\circ}{TM}$ is said to be horizontal if $\delta y^h(X_u) = 0$.

Thus ∇ is metrical along horizontal vectors, in particular along the δ_i's and on the subspace spanned by them, called the horizontal subspace of $T_u \overset{\circ}{TM}$.

The significance of (3.3) is underlined by

Proposition 3.1. *There exists a unique set of* $1-$*forms* $\{\Gamma^i_j\}$ *on* $\overset{\circ}{TM}$ *satisfying*

(a) $d(dx^i) = dx^j \wedge \Gamma^i_j$,

(b) $dg_{ij} = \Gamma^k_i g_{kj} + \Gamma^k_j g_{ik} + 2C_{hij}\delta y^h$.

Proof. The existence was proved in the above. Let $\widetilde{\Gamma}^i_j = \widetilde{\Gamma}^i_{jk} dx^k + \widehat{\Gamma}^i_{jk} dy^k$ be 1−forms satisfying (a) and (b). From $dx^j \wedge \widetilde{\Gamma}^i_j = 0$ it follows that $\widetilde{\Gamma}^i_{jk} = \widetilde{\Gamma}^i_{kj}$ and $\widehat{\Gamma}^i_{jk} = 0$. Subtracting member by member the equations (b) for Γ's and $\widehat{\Gamma}$'s one obtains $(\widetilde{\Gamma}^s_{ik} - \Gamma^s_{ik})g_{sj} + (\widetilde{\Gamma}^s_{jk} - \Gamma^s_{jk})g_{sj} = 0$. Permuting cyclicly the indices i, j, k one gets two new equations which added and subtracting from the result the previous one gives $(\widetilde{\Gamma}^s_{ij} - \Gamma^s_{ij})g_{sk} = 0$. Hence $\widetilde{\Gamma}^s_{ij} = \Gamma^s_{ij}$, q.e.d.

Remark 3.1. As $(\partial_i, \overset{\circ}{\partial}_i)$ is the natural frame on $\overset{\circ}{TM}$, (2.17) is equivalent to

$$\nabla_{\partial_j}\partial_i = \Gamma^k_{ij}\partial_k, \qquad \nabla_{\overset{\circ}{\partial}_j}\partial_i = 0. \tag{3.5}$$

Calculating (3.4) for $(\overset{\circ}{\partial}_h)$ one finds

$$g_{ij}|_h := (\nabla_{\overset{\circ}{\partial}_h} g)(\partial_i, \partial_j) = 2C_{ijh}. \tag{3.6}$$

In Finsler geometry there exist four remarkable Finsler connections which have in common the Cartan nonlinear connection of coefficients $(\overset{c}{N}{}^i_j)$. Among them we have the Rund connection which has the form $(\overset{c}{N}{}^i_j(x,y), F^i_{jk}(x,y), 0)$ with the coefficients $F^i_{jk}(x,y)$ given by

$$F^i_{jk} = \frac{1}{2}g^{ih}(\overset{c}{\delta}_j g_{hk} + \overset{c}{\delta}_h g_{jk}), \tag{3.7}$$

where $\overset{c}{\delta}_j = \partial_j - \overset{c}{N}{}^k_j(x,y)\overset{\circ}{\partial}_k$.

This connection is $h-$metrical but it is not $v-$metrical since by (3.2) we have

$$g_{ij}|_k = 2C_{ijk} \neq 0, \tag{3.6'}$$

except when F^n is a Riemannian space.

Looking at Chern's connection we see that the Γ's from (3.3) coincide with the F's from (3.7) if the (G^i_j) given by (2.16) are just the $(\overset{c}{N}{}^i_j(x,y))$ of Cartan.

This indeed holds as we now prove.

Let $\gamma^i_{jk}(x,y)$ be the "Christoffel symbols"

$$\gamma^i_{jk} = \frac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{jk} - \partial_h g_{jk}).$$

Then the coefficients of the Cartan nonlinear connection are

$$\overset{c}{N}{}^{i}_{j} = \overset{\circ}{\partial}_j(\Gamma^i_{kh} y^k y^h). \tag{3.8}$$

By (2.16) it is sufficient to check that $2G^i = \gamma^i_{kh} y^k y^h$. Equivalently,

$$4G_i = (\partial_j g_{ik} + \partial_k g_{ji} - \partial g_{jk}) y^j y^k. \tag{3.9}$$

By (2.14), $4G_i = y^k \overset{\circ}{\partial}_i \partial_k (F^2) - \partial_i F^2$. Using (2.3) and (2.4), the righthand side of (3.9) becomes

$$2\partial_j (g(iky^k) y^j - \partial_i (g_{jk} y^j y^k) = \partial_j (\overset{\circ}{\partial}_i F^2) y^j - F^2.$$

Hence (3.9) holds.

The equalities $G^i_j = \overset{c}{N}{}^{i}_{j}$, $\Gamma^i_{jk} = F^i_{jk}$, and (3.6) in conjunction with (3.6)', show that we may think of Chern's connection as the Finsler connection $(G^i_j, \Gamma^i_{jk}, 0)$ and furthermore it coincides with the Rund connection.

Since this Finsler connection was first introduced by Chern, it is quite natural that it bear his name. However, Chern has rather graciously suggested that it be called the Chern-Rund connection.

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FINSLER CONNECTIONS IN GENERALIZED LAGRANGE SPACES

by Mihai ANASTASIEI

Abstract

The Chern–Rund connection from Finsler geometry is settled in the generalized Lagrange spaces. For the geometry of these spaces, we refer to [5].

AMS Subject Classification: 53C60.

Key words and phrases: Finsler connections, generalized Lagrange spaces, Chern–Rund connection.

Introduction

In a recent paper, [1], we showed that in a Finsler space the connection introduced by S.S. Chern in 1948 is the same with the connection proposed by H. Rund ten years later and bearing his name. Accordingly, we proposed the name of Rund be replaced with that of Chern, but several geometers including S.S. Chern himself, suggested to call it from now on a Chern–Rund connection.

As S.S. Chern and D. Bao showed in [2], the Chern–Rund connection is very convenient in treating of many global problems in Finsler geometry. This fact determined us to come back to the subject.

The efforts made in defining a covariant derivative and accordingly, a parallel displacement in Finsler space led to a concept generically called a Finsler connection. Among the Finsler connections there exist four, which are remarkable by their properties named the Cartan, Berwald, Chern–Rund and Hashiguchi connections, respectively. These are usually put together in a nice commutative diagram (cf. [3, Ch. III]).

The most utilized is the Cartan connection because it is fully metrical i.e. $h-$ and $v-$metrical, in spite of the fact it has torsion.

But there are some problems involving the Berwald connection which is by no means metrical or the Hashiguchi connection which is only $v-$metrical.

The Chern–Rund connection being $h-$metrical and free of torsion is the nearest to the Levi–Civita connection a fact which explains its adequacy for global problems in Finsler geometry.

The Finsler connections are also suitable for the geometries more general than the Finslerian one as the Lagrange geometry or generalized Lagrange geometry. Our purpose is to review Finsler connections and to settle the Chern–Rund connection in this more general framework.

First, we give in §1 a definition of Finsler connection by local components and introduce its compatibility with a generalized Lagrange metric. Then, in §2, a Finsler connection is defined as a pair (N, ∇), where N is a nonlinear connection on TM and ∇ is a linear connection in the pull–back bundle $c^{-1}TM \longrightarrow TM$ with $\tau : TM \longrightarrow TM$, the tangent bundle over a manifold M. These definitions are equivalent. The four remarkable connections mentioned above are characterized. A special attention is paid to the possibility of determining N from ∇.

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1 Finsler connections. A definition by local components

Let M be a smooth i.e. C^∞ manifold of finite dimension n and $\tau : TM \to M$ its tangent bundle. A local chart $(U, (x^i))$ on M induces a local chart $(\tau^{-1}(U), (x^i, y^i))$ on TM, where $x^i \equiv x^i \circ \tau$ and (y^i) are provided by $u = y^i \left.\dfrac{\partial}{\partial x^i}\right|_p$, $p = \tau(u)$.

A change of coordinates $(x^i, y^i) \longrightarrow (\tilde{x}^i, \tilde{y}^i)$ on TM has the form

$$\tag{1.1} \begin{aligned} &\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n), \ \operatorname{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(x) y^j. \end{aligned}$$

The indices $i, j, k, ...,$ will run from 1 to n and Einstein's convention on summation is implied.

Let $L : TM \longrightarrow R$ be a scalar function on TM. Then $\widetilde{L}(\tilde{x}(x), \tilde{y}(y)) = L(x, y)$, from which, taking partial derivatives and using (1.1), one gets

$$\tag{1.2} \frac{\partial L}{\partial y^i} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial L}{\partial \tilde{y}^k},$$

$$\tag{1.3} \frac{\partial L}{\partial x^i} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial L}{\partial \tilde{x}^k} + \frac{\partial^2 \tilde{x}^k}{\partial x^j \partial x^i} y^j \frac{\partial L}{\partial \tilde{y}^k}.$$

According to (1.2), the set of functions $\left(\dfrac{\partial L}{\partial y^i}(x, y)\right)$ may be regarded as the components of a covector field on TM. From (1.2), it follows that $\left(\dfrac{\partial^2 L}{\partial y^i \partial y^j}(x, y)\right)$ may be also viewed as the components of a (symmetric) tensor field on TM. Thus on TM there exist geometrical objects whose law of transformation under (1.1) is the same as of the corresponding objects

on M. These were called $d-$objects (d is from distinguished) in [5], Finsler objects in [3] and sometimes $M-$objects.

The geometry of $d-$objects is essentially involved in the study of those metrical structures which are more general than Riemannian structures i.e. Finsler structures, Lagrange structures, generalized Lagrange structures (see [5]).

Coming back to (1.3), we see that the behavior of the operators $\dfrac{\partial}{\partial x^i}$ is drastically different from that of $\dfrac{\partial}{\partial y^i}\cdot$ Let us introduce a correction of $\dfrac{\partial}{\partial x^i} := \partial_i$,

$$\delta_i L = \partial_i L + N_i^k(x,y)\dot{\partial}_k,\ \dot{\partial}_k := \frac{\partial}{\partial y^k}, \tag{1.4}$$

such that, with respect to (1.1):

$$\delta_i L = \frac{\partial \tilde{x}^k}{\partial x^i}\,\tilde{\delta}_k L, \tag{1.5}$$

i.e. $(\delta_i L)$ to appear as the components of a covector field on TM. Then the functions $(N_i^k(x,y))$ have to satisfy

$$\frac{\partial \tilde{x}^j}{\partial x^i}\,\tilde{N}_j^h = N_i^j\,\frac{\partial \tilde{x}^h}{\partial x^j} + \frac{\partial^2 \tilde{x}^h}{\partial x^i \partial x^j}\,y^j. \tag{1.6}$$

Note that $(N_i^j(x,y))$ are not the components of a $(1,1)$–tensor field on TM but the difference of two sets of this type is so.

As it is well–known, when M is paracompact, there exists on M a linear connection, say of local coefficients $(\Gamma^i_{jk}(x))$. Then $N^i_k(x,y) = \Gamma^i_{jk}(x)y^j$ verify (1.6). This example assures also the existence of a nonlinear connection within a generally accepted hypothesis on M.

The local vector fields (δ_i), $i = 1,2,...,n$, given by (1.4) are linearly independent and in a point $u \in TM$ they span an $n-$dimensional subspace H_uTM of T_uTM.

Let $\tau_{*,u}$ be the tangent mapping (the Jacobian) of τ. Then $V_uTM = \ker \tau_{*,u}$ is called the vertical subspace of T_uTM. A vertical vector is of the form $X^k(x,y)\dot{\partial}_k$ such that under (1.1) one has

$$\widetilde{X}^k = \frac{\partial x^k}{\partial x^i}\,X^i. \tag{1.7}$$

We immediately have

$$T_uTM = V_uTM \oplus H_uTM. \tag{1.8}$$

Furthermore, $\tau_{*,u}$ restricted to H_uTM gives an isomorphism of it with $T_{\tau(u)}M$ such that $\tau_{*,u}(\delta_i) = \partial_i\big|_{\tau(u)}$.

Conversely, if a supplement of V_uTM in T_uTM is specified by a basis (δ_i), $i = 1, 2, ..., n$, which is carried by τ_* to (∂_i), then letting $\delta_i = \partial_i - N_i^k \dot{\partial}_i$, the condition $\delta_i = \dfrac{\partial \tilde{x}^k}{\partial x^i} \widetilde{\delta}_k$ implies (1.6) for (N_i^k). One says that $(N_i^k(x, y))$ are the coefficients of a nonlinear connection.

A reason for this term is that when (N_i^k) are linear with respect to (y) i.e. $N_i^k(x, y) = G_{ji}^k(x) y^j$, then (G_{ji}^k) are the coefficients of a linear connection on M.

Summarizing the foregoing discussion we may formulate the following two equivalent definitions for a nonlinear connections.

Definition 1.1. A *nonlinear connection* is a set of functions $(N_j^i(x, y))$ defined on each domain of local chart on TM such that an overlaps, (1.6) holds good.

Definition 1.2. A *nonlinear connection* is a smooth distribution $u \longrightarrow H_uTM$ supplementary to the vertical distribution $u \longrightarrow V_uTM$ i.e. (1.7) holds good for every $u \in TM$.

Let $(v^i(x, y))$ be the components of a $d-$vector field. Then $\left(\dfrac{\partial v^i}{\partial y^j}(x, y)\right)$ are the components of a $d-$tensor field of type $(1, 1)$. In other words the partial derivatives with respect to (y^i) are covariant. However, in some circumstances, these have to be replaced by

$$v^i{}_{|j} = \frac{\partial v^i}{\partial y^j} + C_{kj}^i(x, y) v^k, \tag{1.9}$$

where $(C_{kj}^i(x, y))$ are the components of a $d-$tensor field. One of them is as follows.

First, we introduce

Definition 1.3. A $d-$tensor field of type $(0, 2)$ of components $(g_{ij}(x, y))$ which is

a) symmetric, i.e. $g_{ij} = g_{ji}$,

b) nondegenerate i.e. $\det(g_{ij}(x, y)) \neq 0$ and

c) the quadratic form $g_{ij}(x, y)\xi^i xi^j$ $(\xi \in \mathbb{R}^n)$

has constant signature is called a *generalized Lagrange metric* ($GL-$metric for brevity).

Extending (1.9), the covariant derivative of (g_{ij}) is given by

$$g_{ij}\Big|_k = \partial_j v^i - C_{ik}^h g_{hj} - C_{jk}^h g_{ih}. \tag{1.10}$$

One says that the $GL-$metric $(g_{ij}(x, y))$ is $v-$covariant constant if $g_{ijg|k} = 0$.

For the general $v_{|j}^i$, the condition $g_{ij|k} = 0$ can be fulfilled with

$$\overset{c}{C}{}_{ij}^{h} = \frac{1}{2} g^{hk} (\dot{\partial}_i g_{kj} + \dot{\partial}_j g_{ik} - \dot{\partial}_k g_{ij}). \tag{1.11}$$

The partial derivatives with respect to (x^i) are far to be covariant derivatives. A correction of them could be $\partial_j v^i + H^i_{kj}(x,y)v^k$, but $(H^i_{kj}(x,y))$ have a complicated law of transformation A better one is

$$v^i_{|j} = \delta_j v^i + F^i_{kj}(x,y)v^k, \tag{1.12}$$

since then $(F^i_{kj}(x,y))$ changes under (1.1) as the local coefficients of a linear connection on M. These derivatives can be extended to any $d-$tensor field. For instance, the $v-$covariant derivative of $(g_{ij}(x,y))$ is given by (1.10) and its $h-$covariant derivative is

$$g_{ij|k} = \delta_k g_{ij} - F^h_{ik}g_{hj} - F^h_{jk}g_{ih}. \tag{1.13}$$

The $GL-$metric $(g_{ij}(x,y))$ is said to be $h-$covariant constant if $g_{ij|h} = 0$. It is easy to check that the equation $g_{ij|h} = 0$ is satisfied with

$$\overset{c}{F}{}^k_{ij} = \frac{1}{2}g^{kh}(\delta_i g_{hj} + \delta_j g_{ih} - \delta_h g_{ij}). \tag{1.14}$$

The foregoing discussions suggest

Definition 1.4 A Finsler connection is a triad $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$, where $N^i_j(x,y)$ are the coefficients of a nonlinear connection, $F^i_{jk}(x,y)$ are like the coefficients of a linear connection on M and $C^i_{jk}(x,y)$ are the components of a $d-$tensor field.

We have also got a first example of Finsler connection $C\Gamma = (N^i_j(x,y), \overset{c}{F}{}^i_{jk}(x,y), \overset{c}{C}{}^i_{jk}(x,y))$.

Definition 1.5 Let $F\Gamma$ be a Finsler connection and $(g_{ij}(x,y))$ a $GL-$metric. $F\Gamma$ is said to be $h-$metrical if $g_{ij|h} = 0$, $v-$metrical if $g_{ij|h} = 0$ and metrical if the both equations hold.

In the above we have proved

Proposition 1.1 *The Finsler connection $C\Gamma$ is metrical.*

The following $d-$tensor fields are called the torsions of $F\Gamma$:

$$\begin{aligned} T^i_{jk} &= F^i_{jk} - F^i_{kj}, \quad & R^i_{jk} &= \delta_k N^i_j - \delta_j N^i_k, C^i_{jk}, \\ P^i_{jk} &= \dot{\partial}_k N^i_j - F^i_{kj}, \quad & S^i_{jk} &= C^i_{jk} - C^i_{kj}. \end{aligned} \tag{1.15}$$

Remark. R^i_{jk} is the integrability tensor of the horizontal distribution. It measures also the curvature of the nonlinear connection N.

The $d-$tensor fields

$$D^i_j = F^i_{kj}y^k - N^i_j,\ d^i_j = \delta^i_j + C^i_{kj}y^k, \tag{1.16}$$

where (δ^i_j) is Kronecker' symbol, are called $h-$deflection and $v-$deflection of $F\Gamma$, respectively.

From (1.6) we infer that $G^i_{jk} = \dot\partial_j N^i_k$ transform under (1.1) as F^i_{jk}. Thus $B\Gamma = (N^i_j, G^i_{jk}, 0)$ is a Finsler connection. It will be called the Berwald connection. This connection is no $v-$metrical nor $h-$metrical and is free of torsions if and only if N is integrable ($R^i_{jk} = 0$) and symmetric ($\dot\partial_j N^i_k = \dot\partial_k N^i_j$).

The connection $C\Gamma$ will be called the Cartan connection. It is $h-$metrical, $h-$symmetric ($\overset{c}{F}{}^i_{jk}(x,y) = \overset{c}{F}{}^i_{kj}(x,y)$), $v-$metrical and $v-$symmetric. The Finsler connection $H\Gamma = (N^i_j, G^i_{jk}(x,y), \overset{c}{C}{}^i_{kj}(x,y))$ will be called the Hashiguchi connection. This is $v-$metrical, no $h-$metrical and has torsion. The Finsler connection $CR\Gamma = (N^i_j, \overset{c}{F}{}^i_{jk}(x,y), 0)$ will be called the Chern–Rund connection. This is $h-$metrical but not $v-$metrical.

Summarizing, for a fixed nonlinear connection N and a $GL-$metric $(g_{ij}(x, y))$ we have four typical Finsler connections: $B\Gamma$, $C\Gamma$, $H\Gamma$ and $CR\Gamma$.

Let us replace TM by $T_0M = TM \setminus 0$.

A $GL-$metric $(g_{ij}(x,y))$ on T_0M reduces to a Finsler metric if there exists a fundamental Finsler function $F : T_0M \longrightarrow R_+$ such that $g_{ij}(x,y) = \frac{1}{2}\dot\partial_i\dot\partial_j F^2(x,y)$. Taking as N the Cartan nonlinear connection of coefficients $\overset{c}{N}{}^i_j = \frac{1}{2}\dot\partial_j\gamma^i_{oo}$, $\gamma^i_{oo} = \gamma^i_{jk}y^jy^k$, $\gamma^i_{jk} = \frac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk})$, the afore mentioned Finsler connections reduce to the four remarkable connections in Finsler geometry ([3, Ch. III]).

The form of D^i_j in (1.6) shows that one may associate to any $F\Gamma$ a new Finsler connection $(F^i_{kj}y^k - D^i_j,\ F^i_{kj},\ C^i_{kj})$ whose $h-$deflection is just D^i_j, when this is prescribed. In particular, for $D^i_j = 0$ a Finsler connection without $h-$deflection is obtained. In Finsler geometry $B\Gamma$, $C\Gamma$, $H\Gamma$ and $CR\Gamma$ are $h-$deflection free. So we have an explanation why the nonlinear connection was noted quite late in Finsler geometry.

2 Another definition of Finsler connections

Let be $\tau^{-1}TM = \{(u,v) \in TM \times TM, \tau(u) = \tau(v)\}$ fibered over TM by $\pi(u,v) = u$. The local fiber in (u,v) is $T_{\tau(u)}M$. A section in $(\tau^{-1}TM, \pi, TM)$ is locally of the form $\bar X = \bar X^i(x,y)\bar\partial_i$ with $(\bar\partial_i)$ the natural basis in $T_{\tau(u)}M$. It follows that under (1.1) we have

$$\tilde{\bar X}^i = \frac{\partial \tilde x^i}{\partial x^k}\bar X^k. \tag{2.1}$$

$\bar X$ will be called a $\tau-$vector field on TM. It can be identified with the $d-$vector field $(\bar X^i(x,y))$. More general, the tensorial algebra of the pull–back bundle $\tau^{-1}TM$ can be thought of as algebra of $d-$tensor fields on TM. There exists a remarkable $\tau-$vector field $\mathbb{C} : u \longrightarrow (u,u)$, which locally is $y^i\bar\partial_i$ and so it can be identified to the Liouville vector field $\mathbb{C} = y^i\dot\partial_i$.

Theorem 2.1. *There exists a one–to–one correspondence between the set of Finsler connections $F\Gamma$ and the set of pairs (N, ∇) with N a nonlinear connection on TM and ∇ a linear connection in the pull–back bundle $\tau^{-1}TM$.*

Proof. If $F\Gamma$ is specified by $(N^i_j, F^i_{jk}, C^i_{jk})$, we take $N = (N^i_j)$ and define ∇ by

$$\nabla_{\delta_k}\bar{\partial}_i = F^i_{jk}\bar{\partial}_i,\ \nabla_{\dot{\partial}_k}\bar{\partial}_j = C^i_{jk}\bar{\partial}_i. \tag{2.2}$$

In the natural basis ∇ takes the form

$$\nabla_{\partial_k}\bar{\partial}_j = \Gamma^i_{jk}\bar{\partial}_i,\quad \nabla_{\dot{\partial}_k}\bar{\partial}_i = C^i_{jk}\bar{\partial}_i. \tag{2.3}$$

$$\Gamma^i_{jk} = F^i_{jk} + N^h_k C^i_{jh}. \tag{2.4}$$

Conversely, given $N = (N^i_j)$ and ∇ specified by (2.3) it results that $(N^i_j$, F^i_{jk}, $C^i_{jk})$ with F^i_{jk} given by (2.4) is a Finsler connection. □

A GL–metric $(g_{ij}(x, y))$ defines a metrical structure g in the bundle $\tau^{-1}TM$:

$$g = gij(x, y)dx^i \otimes dx^j. \tag{2.5}$$

Conversely, any metrical structure in the bundle $\tau^{-1}TM$ defines by (2.5) a $GL-$metric.

One easily checks

Theorem 2.2 *In the correspondence $F\Gamma \longleftrightarrow (N, \nabla)$ we have*

a) *$F\Gamma$ is h–metrical if and only if $\nabla_{hX}g = 0$,*

b) *$F\Gamma$ is v–metrical if and only if $\nabla_{vX}g = 0$,*

c) *$F\Gamma$ is metrical if and only if $\nabla_X g = 0$,*
for every $X \in \mathcal{X}(TM)$.

Let $\rho : TTM \longrightarrow \tau^{-1}TM$ be the morphism of vector bundles given by $\rho(X_u) = (u, \tau_{*,u}X_u)$, $X_u = T_uTM$, $u \in TM$. It follows that $\ker\rho_u = V_uTM$ i.e. $\rho(\dot{\partial}_i) = 0$ and $\rho(\delta_i) = \bar{\partial}_i$. Alternatively, we may define a morphism $\sigma : TTM \longrightarrow \tau^{-1}TM$ on basis by $\sigma(\delta_i) = 0$, $\sigma(\dot{\partial}_i) = \bar{\partial}_i$. We say that

$$\begin{aligned} &\mathbb{T}_\rho(X, Y) = \nabla_X\rho(Y) - \nabla_Y\rho(X) - \rho[X, Y],\\ &\mathbb{T}_\sigma(X, Y) = \nabla_X\sigma(Y) - \nabla_Y\sigma(X) - \sigma[X, Y],\ X, Y \in \mathcal{X}(TM), \end{aligned} \tag{2.6}$$

are torsions of ∇.

The following characterizations of the Finsler connections $B\Gamma$, $H\Gamma$, $CR\Gamma$ and $C\Gamma$ follow.

Theorem 2.3. *In the correspondence $F\Gamma \longleftrightarrow (N, \nabla)$ we have*

a) *$B\Gamma \longleftrightarrow (N, \nabla)$ with $\mathbb{T}_\sigma(hX, vY) = 0,\ \mathbb{T}_\rho(hX, vY) = 0$;*

b) $H\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\sigma(hX,vY)=0,\ \mathbb{T}_\sigma(vX,vY)=0,\ \nabla_{vX}g=0$*;*

c) $CR\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\rho(hX,vY)=0,\ \mathbb{T}_\rho(hX,hY)=0,\ \nabla_{hX}g=0$*;*

d) $C\Gamma \longleftrightarrow (N,\nabla)$ *with* $\mathbb{T}_\rho(hX,vY)=0,\ \mathbb{T}_\sigma(vX,vY)=0,\ \nabla_{X}g=0$*.*

Proof. The local expressions of $\mathbb{T}_\rho$ and $\mathbb{T}_\sigma$ in conjunction with Theorem 2.2 give the desired results.

Now the following question appears. Which conditions should satisfy ∇ in order to determine N such that the pair (N,∇) to correspond to a Finsler connection. An answer is as follows.

Definition 2.1. A linear connection ∇ in the pull–back bundle $\tau^{-1}TM$ is said to be regular if the subspace $\{X_u \mid \nabla_{X_u}\mathbb{C}=0,\ X\in\mathcal{X}(TM)\}$ of T_uTM is supplementary to V_uTM for every $u\in TM$.

By the definition, every regular connection ∇ induces a nonlinear connection N on TM. The pair (N,∇), as we have seen before, corresponds to a Finsler connection $F\Gamma$. This $F\Gamma$ has to be of a particular form. Indeed, one has

Theorem 2.4. *There exists a bijection between the set of regular connections in* $\tau^{-1}TM$ *and the set of Finsler connections* $F\Gamma=(N^i_j,\ F^i_{jk},\ C^i_{jk})$ *satisfying* $D^i_j=0$ *and* $\det(d^i_h)\neq 0$*.*

Proof. Let ∇ be specified by (2.3). Using $N=(N^i_j)$ provided by the regularity of ∇, we define F^i_{jk} as in (2.4). Then $0=\nabla_{\delta_k}\mathbb{C}=(y^jF^i_{jk}-N^i_k)\bar{\partial}_k$ implies $D^i_k=0$. Contracting (2.4) by y^j we get $N^h_k(d^i_h)=y^j\Gamma^i_{jk}$ and as (N^h_k) is specified this equation has to have an unique solution. Hence with necessity $\det(d^i_h)\neq 0$.

Conversely, let (N,∇) be in correspondence with $F\Gamma$. The condition $D^i_j=0$ assures that the subspace $\{X_u|\nabla_{X_u}\mathbb{C}=0,\ X\in\mathcal{X}(TM),\ u\in TM\}$ is contained in the horizontal subspace H_uTM of N. The condition $\det(d^i_k)\neq 0$ implies that this subspace is supplementary to V_uTM. Thus ∇ is regular and the nonlinear connection derived from it coincides with N. □

Let us assume that (g_{ij}) reduces to a Finsler metric on T_0M. Then $C\Gamma$ is characterized by the following Matsumoto's axioms:

$$(*)\qquad T^i_{jk}=0,\ g_{ij|k}=0,\ S^i_{jk}=0,\ g_{ij}\Big|_k=0,\ D^i_j=0.$$

It results $d^i_j=\delta^i_j$.

Combining these with Theorems 2.4 and 2.3, one obtains

Theorem 2.5. *Let* $F^n=(M,F)$ *be a Finsler space. There exists a unique regular connection* ∇ *in* $\pi^{-1}T_0M$ *satisfying the conditions:*

$$\mathbb{T}_\rho(hX,hY)=0,\ \mathbb{T}_\sigma(vX,vY)=0,\ \nabla_Xg=0,\ X,Y\in\mathcal{X}(T_0M)$$

where h *and* v *are projectors of* N *induced by* ∇*.*

We note that ∇ is determined by F only.

According to [5] the Chern–Rund connection in a Finsler space is characterized by the following axioms:

$$T^i_{jk} = 0,\ g_{ij|k} = 0,\ C^i_{jk} = 0,\ D^i_j = 0.$$

We have again $d^i_j = \delta^i_j$. By Theorems 2.3 and 2.4 we have

Theorem 2.6. *Let $F^n = (M, F)$ be a Finsler space. There exists a unique regular connection ∇ in $\pi^{-1}T_0M$ satisfying the conditions:*

$$\mathbb{T}_\rho(hX, hY) = 0,\ \mathbb{T}_\rho(hX, vY) = 0,\ \nabla_{hX}g = 0,\ X, Y \in \mathcal{X}(T_0M)$$

where h and v are projectors of N induced by ∇.

The systems of axioms for $H\Gamma$ and $B\Gamma$ discussed for minimality in [5] give similar results in view of Theorems 2.3 and 2.4.

The Finsler connections may be viewed also as special liner connections on TM or in the Finsler bundle $\pi^{-1}LM$, where LM is the principal bundle of linear frames on M. We refer to [5] and [3], respectively.

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JACOBI FIELDS IN GENERALIZED LAGRANGE SPACES

BY

M. ANASTASIEI and I. BUCATARU

Abstract

We consider the first variation of those curves on tangent manifold TM which have property that are parallel with respect to the canonical metrical connection in a generalized Lagrange space. Accordingly we introduce and study the Jacobi fields on TM. Several particular cases are discussed.

1 Introduction

Among the notions introduced and studied by Prof.Radu Miron, very interesting and useful for applications is that of generalized Lagrange space, GL-space for brevity. This is a pair made up by a smooth manifold M and a generalized Lagrange metric, shortly a GL-metric. Roughly speaking a GL-metric is a metrical structure in the vertical bundle over the manifold TM. Viewing in local coordinates one can see that its definition was tailored after the basic properties of a Finsler metric. Thus a GL-space appears as a very large generalization of a Finsler space. However, this notion preserves many properties of a Finsler space, the existence of a canonical metrical connection being an important one. The autoparallel curves of this connection are remarkable since in the Finslerian framework these are projecting on the geodesics of the Finsler metric. Calling then also geodesics, we consider their first variations, in Section 3, and accordingly we find a Jacobi equation whose solutions are called Jacobi fields. Some properties of these are found. Next, in Section 4, we consider horizontal and vertical Jacobi fields and we investigate some particular cases. The Section 2 is devoted to some preliminaries and notations.

We express our hearty thanks to Prof.Radu Miron for his constant encouragements and valuable suggestions for our researches along many years.

2 Generalized Lagrange spaces

Let M be a real, smooth i.e. C^∞ manifold of finite dimension n and TM its tangent manifold projected to M by the mapping τ. Set $\overset{\circ}{TM} = TM \setminus \{0_x \in T_xM, x \in M\}$. Let $(U,(x^i))$ be a local chart on M. The indices $i, j, k, ...$will run from 1 to n and the Einstein convention on summation will be implied. Associate to $v \in \tau^{-1}(U)$ the coordinates $x = (x^i(\tau(u)))$ and $y = (y^i)$, provided by $V_{\tau(u)} = y^i\partial_i$, $\partial_i = \dfrac{\partial}{\partial x^i}$, and TM becomes a smooth orientable manifold. A change of coordinates $(x^i, y^i) \mapsto (x^{i'}, y^{i'})$ on TM is as follows:

$$x^{i'} = x^{i'}(x^1, x^2, ..., x^n), \quad y^{i'} = (\partial_j x^{i'})y^j, \quad \mathrm{rank}(\partial_j x^{i'}) = \mathrm{n}. \tag{2.7}$$

Definition 2.1 A set of matrices $(g_{ij}(x,y))$ defined on $\tau^{-1}(U)$ for any open set U in a smooth atlas on M is said to be a GL-metric if
1. $g_{ij}(x,y) = g_{ji}(x,y)$,
2. $g_{ij}(x,y) = (\partial_i x^{k'})(\partial_j x^{h'})g_{h'k'}(x'(x), y'(y))$ on $U \cap U'$,
3. $\det(g_{ij}(x,y)) \neq 0$,
4. The signature of the quadratic form $g_{ij}(x,y)\xi^i\xi^j$, $(\xi^i) \in R^n$ is constant.

The simplest example of a GL-metric is a Riemannian one $\gamma_{ij}(x)$. This is provided according to

$$\gamma_{ij}(x) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}, \text{ with } \mathrm{L} : \overset{\circ}{\mathrm{TM}} \to \mathrm{R} \text{ given by} \tag{2.8}$$

$$L(x,y) = \sqrt{\gamma_{ij}(x)y^iy^j}. \tag{2.9}$$

A little more general GL-metric is a Finslerian one which is provided by (2.2) with a function $L = F : \overset{\circ}{TM} \to R_+$ which is positively homogeneous of degree 1 with respect to y i.e.

$$F(x, \lambda y) = \lambda F(x,y), \quad \lambda > 0. \tag{2.10}$$

A Lagrange metric is a GL-metric provided by (2.2) with any smooth function $L : TM \to R$.

A large class of GL-metrics which are not reducible to the previous ones was considered in [2]:

$$g_{ij}(x,y) = a(x,y)\gamma_{ij}(x,y) + b(x,y)y_iy_j, \tag{2.11}$$

where $\gamma_{ij}(x,y)$ is a Finsler metric, a and b are smooth functions on $\overset{\circ}{TM}$ such that $a(x,y) > 0$, $b(x,y) \geq 0$ and $y_i = \gamma_{ij}y^j$. Particular forms of these GL-metrics were studied in Chapters 11 and 12 of the monograph [4].

3 Jacobi fields

Let us consider, together with a GL-metric $(g_{ij}(x,y))$, a nonlinear connection $(N^i_j(x,y))$. We have the decomposition

$$X = hX + vX \quad \text{for every} \quad \mathrm{X} \in \chi(\mathrm{TM}). \tag{3.1}$$

Denote by P the almost product structure provided by the horizontal and vertical distributions according to

$$P(hX) = hX, \quad P(vX) = -vX. \tag{3.2}$$

Consider also the almost complex structure F defined as follows:

$$F(hx) = -vX, \quad F(vX) = hX. \tag{3.3}$$

Next, setting $G = g_{ij}(x,y)dx^i \times dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j$, $\delta y^i = dy^i + N^i_k(x,y)dx^k$, one gets a metrical structure on TM which is Riemannian if (g_{ij}) is positive definite.

Theorem 3.1 ([4]) *There exists a unique linear connection D on TM with the properties: $DP = O$, $DF = 0$, $DG = 0$ and $hT(h\cdot, h\cdot) = 0$, $vT(v\cdot, v\cdot) = 0$, where T denotes its torsion. In the basis $(\delta_i, \dot{\partial}_i)$, $\delta_i = \partial_i - N^k_i\, \dot{\partial}_k$, this connection is as follows:*

$$\begin{aligned} &D_{\delta_k}\delta_j = L^i_{jk}\delta_i, \quad D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i, \\ &D_{\delta_k}\, \dot{\partial}_j = L^i_{jk}\, \dot{\partial}_i, \quad D_{\dot{\partial}_k}\, \dot{\partial}_j = C^i_{jk}\, \dot{\partial}_i \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} L^i_{jk}(x,y) &= \frac{1}{2}g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}), \\ C^i_{jk}(x,y) &= \frac{1}{2}g^{ih}(\dot{\partial}_j\, g_{hk} + \dot{\partial}_k\, g_{jh} - \dot{\partial}_h\, g_{jk}). \end{aligned} \tag{3.5}$$

We note that D has torsion since the other three components of T do not vanish. We have

$$\begin{aligned} &vT(\delta_k, \delta_j) = R^i_{jk}\delta_i, \quad R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} \\ &hT(\partial_k, \delta_j) = C^i_{jk}\delta_i, \\ &vT(\partial_k, \delta_j) = P^i_{jk}\partial_i, \quad P^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - L^i_{kj}. \end{aligned} \tag{3.6}$$

Thus connection is different from the Levi-Civita connection of G since its torsion does not vanish. We call geodesics the autoparallel curves on

TM with respect to D. Consider a geodesic $c : [0,1] \to TM$ such that $c([0,1]) \subset \tau^{-1}(U)$, where (U, x^i) is a local chart on M. Thus the equation of c is

$$(3.7) \qquad \begin{cases} x^i = x^i(t) \\ y^i = y^i(t), \quad t \in [0,1] \end{cases}$$

The tangent vector field is $\dot{c}(t) = \dfrac{dx^i}{dt}\dfrac{\partial}{\partial x^i} + \dfrac{dy^i}{dt}\dfrac{\partial}{\partial y^i}$. It can be written in the form

$$(3.8) \qquad \dot{c}(t) = \frac{dx^i}{dt}\delta_i + \left(\frac{dy^i}{dt} + N^i_k(x(t), y(t))\frac{dx^k}{dt}\right)\dot{\partial}_i.$$

It results that $\dot{c}(t)$ is a horizontal vector field if and only if

$$(3.9) \qquad \frac{\delta y^i}{dt} = \frac{dy^i}{dt} + N^i_k(x(t), y(t))\frac{dx^k}{dt} = 0.$$

When this condition holds at we say that c is a horizontal geodesic. By (3.8), $\dot{c}(t)$ is a vertical vector field if and only if $x^i = x^i_0$ (constant) i.e. the curve c is in the tangent space $T_{p_0}M$, $p_0 = (x^i_0)$. In this case we say that c is a vertical geodesic. The condition $D_{\dot{c}(t)}\dot{c}(t) = 0$ takes locally the form:

$$(3.10) \qquad \begin{cases} \dfrac{d^2x^k}{dt^2} + L^k_{ij}\dfrac{dx^i}{dt}\dfrac{dx^j}{dt} + C^k_{ij}\dfrac{dx^i}{dt}\dfrac{\delta y^j}{dt} = 0 \\[2ex] \dfrac{\delta^2 x^k}{dt^2} + L^k_{ij}\dfrac{dx^i}{dt}\dfrac{\delta y^j}{dt} + C^k_{ij}\dfrac{\delta y^i}{dt}\dfrac{\delta y^j}{dt} = 0, \end{cases}$$

where we have put

$$(3.11) \qquad \frac{\delta^2 y^k}{dt^2} = \frac{d^2 y^k}{dt^2} + N^k_h\frac{d^2x^h}{dt^2} + \frac{dN^k_h}{dt}\frac{dx^h}{dt} = \frac{d}{dt}\left(\frac{\delta y^k}{dt}\right).$$

Remark. The form of equations (3.10) is preserved by the affine transformation $t \mapsto c_1 t + c_0, \quad c_0, c_1 \in R$ of parameter, only.

Remark. If c is a horizontal geodesic, (3.10) reduces to

$$(3.10') \qquad \frac{d^2x^k}{dt^2} + L^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$

while if c is a vertical geodesic it becomes

$$(3.10'') \qquad \frac{d^2y^k}{dt^2} + C^k_{ij}\frac{dy^i}{dt}\frac{dy^j}{dt} = 0$$

Definition 3.1 Let $c : I \to TM, \quad I = [0,1]$ be a geodesic on TM. A first order variation of it is a smooth mapping $\alpha : (-\varepsilon, \varepsilon) \times I \to TM$ such that $\alpha(0,t) = c(t)$ and $\alpha_s(t) = \alpha(s,t)$ is a geodesic for every $s \in (-\varepsilon, \varepsilon), \ \varepsilon \in R$ and $|\varepsilon|$ small.

Let $\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)$ be the natural basis of the tangent space to $A = (-\varepsilon, \varepsilon) \times I$ in the point (s, t). We set

$$\alpha_{*,(s,t)} \left(\frac{\partial}{\partial t}\right)\bigg|_{s=0} = \tau(t), \quad \alpha_{*,(s,t)} \left(\frac{\partial}{\partial s}\right)\bigg|_{s=0} = V(t).$$

The vector field $t \mapsto \tau(t)$ is in fact $\dot{c}(t)$, the tangent vector field to the curve c and the vector field $t \mapsto V(t)$ will be called the variation vector field induced by α.

As $\alpha_{*,(s,t)} \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = [\tau(t), V(t)]$ we infer $[\tau, V] = 0$. Thus $T(\tau, V) = D_\tau V - D_V \tau$ and $R(\tau, V)\tau = D_\tau D_V \tau - D_V D_\tau \tau - D_{[\tau,V]}\tau = D_\tau D_V \tau$ since $D_\tau \tau = 0$ (c is a geodesic). Furthermore, $R(\tau, V)\tau = D_\tau(D_\tau V - T(\tau, V) = D^2_\tau V - D_\tau T(\tau, V)$. Thus V satisfies the following equation

$$D^2_\tau V + R(V, \tau)\tau - D_\tau T(\tau, V) = 0$$

Definition 3.2 It is called Jacobi field along of a geodesic c any vector field X which is solution of the following Jacobi equation:

$$D^2_{\dot{c}(t)} X + R(X, \dot{c}(t))\dot{c}(t) - D_{\dot{c}(t)} T(\dot{c}(t), X) = 0 \tag{3.12}$$

As in the Riemannian framework one proves:

Proposition 3.1 1) *The solution X of the Jacobi equation is uniquely determined by the initial conditions $X(t_0) = X_0$ and $(D_{\dot{c}(t)} X)(t_0) = V_0,\ t_0 \in I$.*
2) *The set of Jacobi fields is a linear space of dimension $4n$.*
3) *The vector fields $\tau : t \mapsto \tau(t)$ and $\widetilde{\tau} : t \mapsto t\dot{c}(t)$ are Jacobi vector fields along the geodesic c.*
4) *Any Jacobi vector field X along c is of the form $X = a\tau + b\widetilde{\tau} + Y$, with a and b constants and Y is a Jacobi vector field which is ortogonal to τ with respect to G.*

4 Some particular cases

The following particular cases have to be considered:
a) c is a horizontal geodesic and X is horizontal.
b) c is a vertical geodesic and X is vertical.

In the case a) we have $T(X, \tau) = T(hX, h\tau) = hT(hX, h\tau) + vT(hX, h\tau) = -v[X, \tau] = 0$ since $[X, \tau] = 0$.

Thus (3.12) reduces to

$$D^2_\tau X + R(X, \tau)\tau = 0. \tag{4.1}$$

We notice that for a Finsler metric of a Finsler space F^n, D is exactly the Cartan connection of F^n. In (4.1), R is the $(hh)h$- curvature of D which,

coincides to the Chern-Rund connection (see [2]). Thus for a Finsler metric the equation (4.1) is nothing but the equation (4.13) in [3]. Taking c as the lift $\left(x^i, \dfrac{dx^i}{dt}\right)$ of a curve $x = x^i(t)$ on M one may try a study similar to that from [3] with some cautions regarding the parameter of the curve $x^i = x^i(t)$.

In case b) we have again $T(X, \tau) = 0$ and (3.12) reduces to

$$D_\tau^2 X + R(X, \tau)\tau = 0. \tag{4.2}$$

In this case R is the $(vv)v$-curvature of D, usually denoted by S. Now the curve c is entirely in $T_{p_0}M$, $p_0 = (x_0^i)$. The space $T_{p_0}M$ has a pseudo-Riemannian structure given by $g_{ij}(x_0, y)$ whose curvature is S. This pseudo-Riemannian structure is not flat except if the GL-metric $g_{ij}(x, y)$ is a Riemannian one. The equation (4.2) is exactly the Jacoby equation for $(T_{p_0}M, g_{ij}(x_0, y))$ and when $g_{ij}(x_0, y)$ is positive defined one may apply the theory from the Riemannian case. The geodesics in $(T_{p_0}M, g_{ij}(x_0, y))$ are sometimes called $v-$paths.

Let us consider the GL-metric

$$g_{ij}(x, y) = \gamma_{ij}(x) + b(x, y)y_i y_j, \tag{4.3}$$

where $\gamma_{ij}(x)$is a Riemannian metric and $b : \overset{\circ}{TM} \to R$ is a smooth function such that $b(x, y) > 0$. Together with this GL-metric we may consider the nonlinear connection $N_j^i(x, y) = \gamma_{kj}^i(x)y^k$, where $\gamma_{kj}^i(x)$ are the Christoffel symbols derived from $(\gamma_{ij}(x))$. It is not difficult to see that, with this choice the projection $\tau(TM, G) \to (M, \gamma)$ is a Riemannian submersion. Thus the general theory of submersion may be used in order to investigate (4.1). It follows that if a curve is horizontal in a point it is horizontal at any points and any horizontal curves is projected by τ on a geodesic of (M, γ). Furthermore, the Jacobi fields on TM which are solutions of (4.1) are projected by τ_* on Jacobi fields on (M, γ).

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THE BEIL METRICS ASSOCIATED TO A FINSLER SPACE

BY

M. ANASTASIEI and H. SHIMADA

1 Introduction

Let $F^n = (M, F)$ be a Finsler space with M a smooth i.e. C^∞ manifold and $F : TM \to R$, $(x, y) \to F(x, y)$. Assume that F^n is endowed with a Finsler 1–form $\beta_i(x, y)$ and set $\beta = \beta_i(x, y)y^i$. Here $i, j, k, ...$ will run from 1 to $n = \dim M$ and the Einstein convention on summation is implied. Then ${}^*F = L(F, \beta)$ in some conditions on L is so that ${}^*F^n = (M, {}^*F)$ is a new Finsler space. It is said that ${}^*F^n$ is obtained from F^n by a β–change [7],[10].

Typical for ${}^*F^n$ are the Randers and Kropina spaces which are obtained from a Riemannian space by particular β–changes.

Let $g_{ij}(x, y)$ be the Finsler metric tensor of F^n. If one wishes the construction of a new Finsler metric ${}^*g_{ij}$ which depends on $g_{ij}(x, y)$, then because of the linear structure of the set of Finsler tensor fields of a given type, the most general choice is

$$
{}^*g_{ij}(x, y) = \rho(x, y)g_{ij}(x, y) + \sigma(x, y)B_{ij}(x, y), \tag{1.1}
$$

for ρ and σ two Finsler scalars and $B_{ij}(x, y)$ a symmetric Finsler tensor field of type $(0, 2)$. We may say that ${}^*g_{ij}$ is obtained from g_{ij} by a B–change.

It is clear that ${}^*g_{ij}$ is no longer a Finsler metric except if some strong conditions on ρ, σ and B_{ij} are imposed. Metrics similar to (1.1) appear in [2] and [5] from physical considerations. See also [11].

In order to relax such conditions we do not ask ${}^*g_{ij}$ be a Finsler metric but a generalized Lagrange metric in Miron' sense, shortly a GL–metric. For the theory of the GL–metrics we refer to [9], ch.X.

As such $({}^*g_{ij})$ has to satisfy

a) $\det({}^*g_{ij}) \neq 0$ and

b) The quadratic from ${}^*g_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in \mathbb{R}^n$, to be of constant signature.

Even this minimal requirements are not easy to be fullfiled except for some particular σ, ρ and B_{ij}.

By our best knowledge the following two particular forms of the GL–metric (1.1) were studied

$$ {}^{*}g_{ij}(x,y) = e^{2\alpha(x,y)} g_{ij}(x,y). \tag{1.2} $$

This class of GL–metrics contains the Miron–Tavakol metrics used by them in General Relativity and the Antonelli metrics which were introduced by P.L. Antonelli for some studies in Biology and Ecology. For details see [9], ch.XI, and reference therein.

$$ {}^{*}g_{ij}(x,y) = g_{ij}(x,y) + \sigma(x,y) y_i y_j,\ \ y_i = g_{ij}(x,y) y^j. \tag{1.3} $$

Particular forms of the GL–metric (1.3) were used by R. Miron in Relativistic Geometrical Optics. See also [9], ch.XII.

Some particular forms of the GL–metric

$$ {}^{*}g_{ij}(x,y) = g_{ij}(x,y) + \sigma(x,y) B_i(x,y) B_j(x,y), \tag{1.4} $$

with $B_i(x,y) = g_{ij}(x,y) B^j(x,y)$ for $B^j(x,y)$ a given Finsler vector field were introduced by R.G. Beil in order to develop his interesting unified field theory ([4]). These were called Beil metrics. As such we refer to ${}^{*}g_{ij}$ in (1.4) as to the Beil metric, too. The following comment of R.G. Beil is illuminating on (1.4). "Since in my unified theory the quantity k which correspond to your σ is related to the gravitational constant, this means that a possible physical interpretation of your theory with a $y-$dependent σ is that gravitation itself is velocity dependent. This possibility is mentioned, for example, in Section 40.8 of the famous book *Gravitation* by Misner, Thorne and Wheeler". See [13].

The particular form of (1.4) obtained for $\sigma = 1$ and $B_i = \dfrac{\partial f}{\partial x^i}$, $f : M \to \mathbb{R}$ was considered by C. Udrişte in [14]. He proved that if f is proper i.e. $f^{-1}(K)$ is a compact set whenever K is compact, then the Finsler manifold $(M, {}^{*}g_{ij}(x,y))$ is complete. A Riemannian version of (1.1), that is, was used by T. Aubin in order to prove that any compact Riemannian manifold of dimension greater then 2 admits a metric whose scalar curvature is a negative constant. See [3] and for other connected results.

The geometry of the GL–metrics (1.4) was not investigated in a systematic way. It is our purpose to fill this gap. After some preliminaries in Section 2, we show in Section 3 that $({}^{*}g_{ij})$ from (1.4) is a GL–metric and we point out cases when it reduces to a Lagrange or to a Finsler metric. In Section 4 we discuss possibilities for introducing metrical connections for the GL–space $(M, {}^{*}g_{ij})$. In Section 5 we digress on parallel and resp. concurrent Finsler vector fields showing that the usual definitions for these notions are also justified from the viewpoint of the almost Hermitian model of a GL–space. For such a model see [9], ch. X. Section 6 is devoted to the analysis of the GL–metric (1.4) for B^i a concurent Finsler vector field. For σ a constant we rediscover a modification of a Finsler function studied by M. Matsumoto and K. Eguchi in [8]. The case when σ is a solution of the so–called Tavakol–Van der Berg equation is investigated, too. In Section 7 we treat a Beil metric associated to a Finsler space with (α, β)–metric. It is a future task to find properties of the GL–metric (1.4) when F^n is a particular Finsler space or its dimension is low (2 or 3).

2 Preliminaries

Let M be a smooth i.e. C^∞ manifold, paracompact and of dimension n, TM its tangent manifold and $\tau : TM \to M$ its tangent bundle. If $x = (x^i)$, $i, j, k, ... = 1, ..., n$ are local coordinates on M, then the induced coordinates on TM will be $(x, y) = (x^i : x^i \circ \tau, y^i)$ with (y^i) provided by $u_x = y^i \left.\dfrac{\partial}{\partial x^i}\right|_x$, $u \in T_xM$, $x \in M$. The change of coordinates $(x, y) \to (\tilde{x}, \tilde{y})$ on TM are as follows.

$$\tag{2.1} \begin{aligned} &\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n),\ \text{rank}\left(\frac{\partial \tilde{x}^i}{\partial x^k}\right) = n \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^k}(x) y^k. \end{aligned}$$

The geometrical objects on TM whose local components change by (2.1) as on M i.e. ignoring their dependence on y, will be called Finsler objects as in [7] or d–objects as in [9].

We set $\partial_i := \dfrac{\partial}{\partial x^i}$, $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}$ and notice that the vertical subspace of T_uTM i.e. $V_uTM = \text{Ker}\,(D\tau)_u$, $u \in TM$, where $D\tau$ means the differential of τ, is spanned by $(\dot{\partial}_i)$. The d–objects can be expressed using $(\dot{\partial}_i)$.

A function $F : TM \to \mathbb{R}$ which is positive, smooth on $TM \setminus 0$ and only continuous in the rest, positively homogeneous of degree 1 with respect to y i.e. $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$ and with the quadratic form $g_{ij}(x, y)\xi^i\xi^j$, $(\xi^i) \in \mathbb{R}^n$ nondegenerate and of constant signature, where

$$\tag{2.2} g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2,$$

is called a fundamental Finsler function. The pair $F^n = (M, F)$ is called a Finsler space.

The function $g_{ij}(x, y)$ are the components of a Finsler tensor field called the Finsler metric of F^n.

A supplement H_uTM of V_uTM i.e. the decomposition in a direct sum $T_uTM = H_uTM \oplus V_uTM$ holds, will be called the horizontal space and the distribution $u \to H_uTM$ will be called a horizontal distribution. A basis of it of the form $\delta_i = \partial_i - N_i^k(x, y)\dot{\partial}_k$, provides the functions $(N_i^k(x, y))$ called the local coefficients. These functions have a special rule of change by (2.1) and in turn they completely determine the horizontal distribution called also a nonlinear connection. Then $(\delta_i, \dot{\partial}_i)$ is a basis adapted to the previous decomposition of T_uTM. The Finsler objects may be also expressed by using (δ_i). We notice that (δ_i) are Finsler vector fields. For more details we refer to [7],[9].

3 The Beil metric

Let $F^n = (M, F)$ be a Finsler space and $g_{ij}(x, y)$ its Finsler metric. Assume that F^n is endowed with a Finsler vector field $B = B^i(x, y)\dot{\partial}_i$ and let

$B_i(x,y)dx^i$ the Finsler 1-form with $B_i = g_{ik}B^k$. The lowering and rising of indices will be done with (g_{ij}) and (g^{jk}), where $g^{jk}g_{ki} = \delta_i^j$, respectively. Let $\sigma : TM \to \mathbb{R}$, $(x,y) \to \sigma(x,y)$ a Finsler scalar. We set

$$ {}^*g_{ij}(x,y) = g_{ij}(x,y) + \sigma(x,y)B_i(x,y)B_j(x,y). \tag{3.1}$$

The functions $({}^*g_{ij})$ from (3.1) define for $\sigma > 0$ a positive definite GL–metric called *the Beil metric.*

It is clear that $({}^*g_{ij})$ are the components of a symmetric d–tensor field. We look for the inverse of the matrix $({}^*g_{ij})$ in the form ${}^*g^{jk} = {}^*g^{jk} - {}^*\sigma B^jB^k$ with ${}^*\sigma$ to be determined. From ${}^*g_{ij}{}^*g^{jk} = \delta_i^k$ it follows that ${}^*\sigma = \dfrac{\sigma}{1+\sigma B^2}$, with $B^2 = B_iB^i = g_{ij}B^iB^j$ (the length of B with respect to g_{ij}). Thus we have

$$ {}^*g^{jk} = g^{jk} - \frac{\sigma}{1+\sigma B^2}B^jB^k. \tag{3.2}$$

Consequently, we have $\det(g_{ij}) \neq 0$.

The quadratic from $\Phi(\xi) = {}^*g_{ij}\xi^i\xi^j = g_{ij}\xi^i\xi^j + \sigma(B_k\xi^k)^2$ is clear positive definite in our hypothesis. **q.e.d.**

We notice that (3.2) holds in the weaker condition $\sigma \neq -\dfrac{1}{B^2}$ and if $g_{ij}\xi^i\xi^j$ is only of constant signature, the signature of $\Phi(\xi)$ will be constant for some σ and (B^k) at least locally.

Remark 3.1. The GL–metric (3.1) appears in papers by R.G. Beil ([4]) for F^n a pseudo–Riemannian space or a Minkowski space. It was called Beil's metric.

We notice that for $B^i = y^i$ in (3.1) one obtains a general version of the Synge metric which was used by R. Miron for a geometrical theory of Relativistic Optics (cf. [9], ch.XI).

In the following we shall assume $B^i \neq y^i$ and use the ideas and techniques from [9], ch.XI.

One says that ${}^*g_{ij}$ is reducible to a Lagrange metric, shortly an L–metric if there exists a Lagrangian $L : TM \to \mathbb{R}$ such that ${}^*g_{ij} = \dfrac{1}{2}\dot{\partial}_i\dot{\partial}_jL$. A necessary and sufficient condition for ${}^*g_{ij}$ be reducible to an L–metric is the symmetry in all indices of the Cartan tensor field ${}^*C_{ijk} = \dfrac{1}{2}\dot{\partial}_k{}^*g_{ij}$ i.e.

$$ \dot{\partial}_k{}^*g_{ij} = \dot{\partial}_i{}^*g_{kj}. \tag{3.3}$$

Using (3.1) this condition becomes

$$ \begin{gathered} \dot{\sigma}_kB_iB_j - \dot{\sigma}_iB_kB_j + \sigma(\dot{\partial}_kB_i\cdot B_j - \dot{\partial}_iB_k\cdot B_j) + \\ +\sigma(B_i\cdot\dot{\partial}_kB_j - B_k\cdot\dot{\partial}_iB_j) = 0,\ \dot{\sigma}_k := \dot{\partial}_k\sigma. \end{gathered} \tag{3.4}$$

Multiplying it by B^j one gets

$$B^2(\dot\sigma_k B_i - \dot\sigma_i B_k) + \sigma B^2(\dot\partial_k B_i - \dot\partial_i B_k) + \\ + \sigma(B_i \cdot \dot\partial_k B_j \cdot B^j - B_k \dot\partial_i B_j \cdot B^j) = 0. \tag{3.5}$$

If (3.4) is an identity, then (3.5) should be an identity for any σ and B_i. But for $B_i = B_i(x)$ and $\sigma = F^2$, (3.5) reduces to $y_k B_i - y_i B_k = 0$ which is not an identity for any B_i. Thus in general ${}^*g_{ij}(x,y)$ is not reducible to an L–metric.

We have a case when ${}^*g_{ij}(x,y)$ is an L–metric as follows.

Proposition 3.1. *Assume $B_i = B_i(x)$. If $\sigma(x,y) = f(B_i(x)y^i)$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$, then ${}^*g_{ij}$ is an L–metric.*

Indeed, it is easy to check that in these hypothesis (3.4) identically holds. Notice that we do not know which is L such that ${}^*g_{ij} = \frac{1}{2}\dot\partial_i\dot\partial_j L$.

It is said that ${}^*g_{ij}(x,y)$ is weakly regular if its absolute energy

$$\mathcal{E}(x,y) := {}^*g_{ij}(x,y)y^iy^j = F^2(x,y) + \sigma(x,y)(B_iy^i)^2 \tag{3.6}$$

is a regular Lagrangian i.e. the matrix with the entries

$$a_{kh}(x,y) = \frac{1}{2}\dot\partial_k\dot\partial_h\mathcal{E}, \tag{3.7}$$

is of rank n.

A direct calculation yields

$$a_{kh} = g_{kh} + \frac{1}{2}\dot\sigma_{kh}\beta^2 + \beta(\dot\sigma_k\dot\beta_h + \dot\sigma_h\dot\beta_k) + \sigma\dot\beta_k\dot\beta_h + \sigma\beta\dot\beta_{kh}, \tag{3.8}$$

$$\beta := B_i(x,y)y^i, \dot\beta_k := \dot\partial_k\beta, \dot\beta_{kh} := \dot\partial_k\dot\partial_h\beta, \dot\sigma_{kh} := \dot\partial_k\dot\partial_h\sigma,\ \dot\sigma_k := \dot\partial_k\sigma \tag{3.8$'$}$$

It is hopeless to decide if a_{kh} is invertible or not. However we have some interesting particular cases.

Proposition 3.2

a) *If B is orthogonal to the Liouville vector field $\mathbb{C} = y^i\dot\partial_i$, then ${}^*g_{ij}$ is weakly regular and $a_{kh}(x,y) = g_{kh}(x,y)$.*

b) *If $B_i = B_i(x)$ and $\sigma(x,y) = f(\beta)$ for some smooth function $f : R \to R$, then ${}^*g_{ij}$ is weakly regular if and only if $1+\varphi(\beta)B^2 \neq 0$, where $2\varphi(\beta) = \beta^2 f'' + 4\beta f' + 2f,\ f' = \frac{df}{d\beta},\ f'' = \frac{d^2f}{d\beta^2}$ and we have*

$$a_{kh}(x,y) = g_{kh}(x,y) + \varphi(x,y)B_k(x)B_h(x). \tag{3.9}$$

Proof. a) The condition B orthogonal to $\mathbb{C}$ is equivalent to $\beta = 0$. Thus $\mathcal{E}(x,y) = F^2(x,y)$ and so $a_{kh} = g_{kh}$.

b) By a direct calculation one finds (3.9). Hence (a_{kh}) has the same form as ${}^*g_{kh}$ with σ replaced by φ. The conclusion follows.

We keep the hypothesis $B_i = B_i(x)$ and $\sigma = f(\beta)$, $\beta \neq 0$. From (3.9) we see that we have again $a_{kh} = g_{kh}$ when $\varphi = 0$. The differential equation $\beta^2 f'' + 4\beta f' + 2f = 0$ takes the form $(\beta^2 f' + 2\beta f)' = 0$ and so its general solution is $f(\beta) = \dfrac{a}{\beta} + \dfrac{b}{\beta^2}$, $a, b \in \mathbb{R}$. The metric ${}^*g_{ij}$ becomes

$$ {}^*g_{ij} = g_{ij} + \left(\frac{a}{B_i(x)y^i} + \frac{b}{(B_s(x)y^s)^2} \right) B_i(x)B_j(x). \tag{3.10}$$

Notice that although ${}^*g_{ij}$ is an L–metric, we do not yet know the Lagrangian L.

The absolute energy of ${}^*g_{ij}$ is now $\mathcal{E} = F^2 + a(F_i(x)y^i) + b$ and the Lagrange space $L^n = (M, \mathcal{E})$ is called an almost Finslerian–Lagrange space (see Section 6, ch.IX of [9]).

We may put (3.9) into the form

$$a_{kh}(x,y) = {}^*g_{kh} + \left(\frac{1}{2}\beta^2 f'' + 2\beta f' \right) B_k B_h. \tag{3.9$'$}$$

Thus we see that $a_{kh} = {}^*g_{kh}$ if and only if f is a solution of the differential equation

$$\frac{1}{2} f''\beta^2 + 2\beta f' = 0 \text{ i.e. } f(\beta) = c - \frac{d}{\beta^3},\ c, d \in \mathbb{R}.$$

We know that ${}^*g_{kh}$ is an L–metric (in previous hypothesis). The condition $a_{kh} = {}^*g_{kh}$ gives L in the form $L(x,y) = \mathcal{E}(x,y) + A_i(x)y^i + \psi(x)$, where A_i is a covector and ψ a scalar. Inserting here $\mathcal{E}$ we get

(3.10)$'$

$$L(x,y) = F^2(x,y) + c(B_i(x,y)y^i)^2 - \frac{d}{B_i(x)y^i} + A_i(x)y^i + \psi(x),\ c, d \in \mathbb{R}.$$

Therefore we found a case when ${}^*g_{ij}$ is an L–metric with L of explicit form (3.10)$'$.

Remark 3.2 In the hypothesis of a) in Proposition 3.2, ${}^*g_{ij}$ is not necessarily an L–metric. If $\sigma(x,y)$ and $B_i(x,y)$ are positively homogeneous of degree 0, then ${}^*g_{ij}(x,y)$ is so and $(M, {}^* g_{ij})$ is a generalized Finsler space in Izumi' sense (see [6]).

Remark 3.3. The condition B orthogonal to $\mathbb{C}$ is equivalent with the condition B is tangent to the indicatrix bundle $I(M) \subset TM$.

Caution. The conditions $\beta = 0$ and $B_i = B_i(x)$ are incompatible since they lead to $B = 0$.

Remark 3.4. If in (3.10) we take $d = 0$, $A_i = 0$, $\psi = 0$, $c > 0$, then ${}^*F^2 := L(x, y)$ is positively homogeneous of degree 2 and so ${}^*F^n = (M, {}^*F)$ becomes a Finsler space. Notice that ${}^*F$ is getting from F by a β–change and in this case ${}^*g_{ij}$ reduces to a Finsler metric.

Remark 3.5. An interesting Beil metric can be associated to a Finsler space F^n with an (α, β)–metric. Here $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$, where a_{ij} is a Riemannian metric an b_i a covector field on M. One may consider

$$ {}^*g_{ij}(x, y) = a_{ij}(x) + \sigma(x, y)b_i(x)b_j(x), \tag{3.12}$$

where σ is a Finsler scalar such that $1 + \sigma b^2 \neq 0$ for $b^2 = a^{ij}b_ib_j$. This GL–metric is not reducible to an L–metric or a Finsler metric. The previous discussion applies, too.

4 Metrical connections for $GL = (M, {}^* g_{ij}(x, y))$

In Finsler geometry as well as in their generalizations, the nonlinear connections play an important role. For instance these connections allow us to work with d– or Finsler objects and so to keep and check easily the geometrical meaning of calculation in local coordinates.

A nonlinear connection always exists if M is paracompact. But the nonlinear connections derived from or associated in a way to a GL–metric are much more useful. There are no possibilities to find nonlinear connections for any GL–metric. But there are some classes of GL–metrics for which such possibilities exist. One is that of weakly regular GL–metrics and as it is well known there exist nonlinear connections canonically derived from a Lagrangian, a Finslerian or a Riemannian metric. See [9] for details.

We recall here the Cartan nonlinear connection for F^n. Set

$$\gamma^i_{jk}(x, y) = \frac{1}{2}g^{ih}(\partial_j g_{hk} + \partial_k g_{hj} - \partial_h g_{jk}), \ \gamma^i_{00} := \gamma^i_{jk}y^jy^k. \tag{4.1}$$

Then $\overset{\circ}{N}{}^i_j = \dfrac{1}{2}\dot{\partial}_j\gamma^i_{00}$ are the local coefficients of the Cartan nonlinear connection.

For any Finsler connection $F\Gamma(N)$ we denote by $_{|k}$ and $\big|_k$ its h– and v–covariant derivatives. Then $F\Gamma(N)$ is called h–metrical if $g_{ij|k} = 0$ and v–metrical if $g_{ij}\big|_k = 0$.

We consider

$$\begin{aligned} F^i_{jk} &= \frac{1}{2}g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}), \\ C^i_{jk} &= \frac{1}{2}g^{ih}(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}), \end{aligned} \tag{4.2}$$

where $\delta_j = \partial_j - \overset{\circ}{N}{}^k_j\dot{\partial}_k$. For F^n we have four remarkable Finsler connections based on $(\overset{\circ}{N}{}^i_j)$.

We mention here only the Cartan connection $C\Gamma(\overset{\circ}{N}) = (\overset{\circ}{N}{}^i_j, F^i_{jk}, C^i_{jk})$. This is $v-$ and h–metrical and two torsions of it vanishes.

Let us come back to the GL–metric (3.1). We cannot derive a nonlinear connection from it. But since it is constructed with $g_{ij}(x, y)$, we may take into consideration the Cartan nonlinear connection $(\overset{\circ}{N}{}^i_j)$ and then all possible nonlinear connections have the form $N^i_j = \overset{\circ}{N}{}^i_j - A^i_j$ with $A^i_j(x, y)$ an arbitrary Finsler tensor field of type $(1, 1)$.

Now we replace in the right side of (4.2) the metric g_{ij} by ${}^*g_{ij}$ and the operator δ_j by ${}^s\delta_j = \partial_j - \overset{\circ}{N}{}^k_j\dot{\partial}_k + A^k_j\dot{\partial}_k$ and denote the results in the left side by ${}^sF^i_{jk}$ and ${}^sC^i_{jk}$, respectively. Thus we get a Finsler connection ${}^sC\Gamma(N) = (N^i_j, {}^sF^i_{jk}, {}^sC^i_{jk})$ which we call standard metrical connection of GL.

This connection is metrical i.e. ${}^*g_{\underset{ij|k}{s}} = 0$, ${}^*g_{ij}\overset{s}{\Big|}_k = 0$ and its $h(hh)$–torsion and $v(vv)$–torsion vanish. It is clear that it depends on A^i_j but if A^i_j is given apriori it is the unique Finsler connection with the above properties. For $A^i_j = 0$ we set ${}^*F := {}^sF$ and ${}^*C := {}^sC$. Thus we have

$$
\begin{aligned}
&{}^sF^i_{jk} = {}^*F^i_{jk} + \frac{1}{2}{}^*g^{ih}(A^s_j\dot{\partial}_s{}^*g_{hk} + A^s_k\dot{\partial}_s{}^*g_{hj} - A^s_h\dot{\partial}_s{}^*g_{jk})\\
&{}^sC^i_{jk} = {}^*C^i_{jk}.
\end{aligned}
\tag{4.3}
$$

The first equation in (4.3) takes also the form

$$
{}^sF_{jik} = {}^*F_{jik} + {}^*C_{kis}A^s_j + {}^*C_{jis}A^s_k - {}^*g^{ih}A^l_h{}^*C_{jkl}.
$$

Remark 4.1. If $({}^*g_{ij})$ reduces to an L–metric or to a Finsler metric, (4.3) becomes

$$
\begin{aligned}
&{}^sF^i_{jk} = {}^*F^i_{jk} + C^i_{ks}A^s_j\\
&{}^sC^i_{jk} = {}^*C^i_{jk}.
\end{aligned}
\tag{4.3$'$}
$$

We notice the following possible choices of A^i_j : $\lambda(x, y)\delta^i_j$, y^iy_j, B^iy_j, y^iB_j, B^iB_j.

By (3.1) we find

$$
\begin{aligned}
{}^*F^i_{jk} &= B^i_sF^s_{jk} + \frac{\sigma}{2}\overset{*}{g}{}^{ih}[\delta_j(B_hB_k) + \delta_k(B_hB_j) - \delta_h(B_jB_k)]+\\
&+\frac{1}{2}{}^*g^{ih}(\sigma_jB_hB_k + \sigma_kB_hB_j - \sigma_hB_jB_k),\\
{}^*C^i_{jk} &= B^i_sC^s_{jk} + \frac{\sigma}{2}\overset{*}{g}{}^{ih}[\dot{\partial}_j(B_hB_k) + \dot{\partial}_j(B_hB_j) - \dot{\partial}_j(B_jB_k)]+\\
&+\frac{1}{2}{}^*g^{ih}(\dot{\sigma}_jB_hB_k + \dot{\sigma}_kB_hB_j - \dot{\sigma}_hB_jB_k), \quad \text{with}
\end{aligned}
\tag{4.4}
$$

$$
B^i_s = \dot{\partial}^i_s - {}^*\sigma B^iB_s,\ \sigma_k := \delta_k\sigma,\ \dot{\sigma}_k := \dot{\partial}_k\sigma,\ {}^*\sigma = \sigma/(1 + \sigma B^2).
\tag{4.4$'$}
$$

Now, ${}^sF^i_{jk}$ and ${}^sC^i_{jk}$ are easily deduced from (4.3).

Remark 4.2. The matrix B^i_s is invertible. Its inverse is $(B^{-1})^s_k = \delta^s_k + \sigma B^s B_k$. As such from (4.4) we can find F and C as depending on ${}^*F$ and ${}^*C$.

In order to evaluate the torsions and curvatures of ${}^*C\Gamma({}^cN)$ it is more convenient to put (4.4) into the form

$$\begin{aligned} {}^*F^i_{jk} &= F^i_{jk} + \Lambda^i_{jk}, \\ {}^*C^i_{jk} &= C^i_{jk} + \overset{\circ}{\Lambda}{}^i_{jk}, \text{ for} \end{aligned} \tag{4.5}$$

$$\begin{aligned} \Lambda^i_{jk} &= \frac{1}{2}{}^*g^{ih}[\delta_k(\sigma B_j B_h) + \delta_j(\sigma B_h B_k) - \delta_h(\sigma B_j B_k)] + \\ &\quad - {}^*\sigma B^i B^h F_{jhk} \\ \overset{\circ}{\Lambda}{}^i_{jk} &= \frac{1}{2}{}^*g^{ih}[\dot\partial_k(\sigma B_j B_h) + \dot\partial_j(\sigma B_h B_k) - \dot\partial_h(\sigma B_j B_k)] + \\ &\quad - {}^*\sigma B^i B^h C_{ijk}. \end{aligned} \tag{4.5$'$}$$

The torsions of ${}^*C\Gamma({}^cN)$ are as follows.

$$\begin{aligned} &{}^*T^i_{jk} = 0,\ {}^*R^i_{jk} = R^i_{jk},\ {}^*S^i_{jk} = 0 \\ &{}^*P^i_{jk} = P^i_{jk} - \Lambda^i_{kj} \text{ and } {}^*C^i_{jk} \text{ from (4.5).} \end{aligned} \tag{4.6}$$

As for the curvatures we have

$${}^*S_j{}^i{}_{kh} = S_j{}^i{}_{kh} + \overset{\circ}{\Lambda}{}_j{}^i{}_{kh} + (C^s_{jk}\overset{\circ}{\Lambda}{}^i_{sh} + \overset{o}{\Lambda}{}^s_{jk}C^i_{sh} - (k/h)) \tag{4.7}$$

$$\overset{\circ}{\Lambda}{}_j{}^i{}_{kh} = \dot\partial_h \overset{\circ}{\Lambda}{}^i_{jk} + \overset{\circ}{\Lambda}{}^s_{jk}\overset{o}{\Lambda}{}^i_{sh} - (k/h), \tag{4.7$'$}$$

where $-(k/h)$ means the substraction of the preceeding terms with k replaced by h.

$$\begin{aligned} {}^*P_j{}^i{}_{kh} &= P_j{}^i{}_{kh} + \dot\partial_h\Lambda^i_{jk} - \overset{\circ}{\Lambda}{}^i_{jh|k} - C^i_{jh\|k} - \overset{\circ}{\Lambda}{}^i_{jh\|k} + \\ &+ \dot\partial_k C^i_{jh} + \dot\partial_k \overset{o}{\Lambda}{}^i_{jh} - C^i_{js}\Lambda^s_{hk} + \overset{o}{\Lambda}{}^i_{js}P^s_{hk} - \overset{o}{\Lambda}{}^i_{js}\Lambda^s_{kh}, \end{aligned} \tag{4.8}$$

where ${}_{\|}k$ denotes a covariant derivative constructed with Λ^i_{jk}.

$${}^*R_j{}^i{}_{kh} = R_j{}^i{}_{kh} + \Lambda_j{}^i{}_{kh} + (F^s_{jk}\Lambda^i_{sh} + \Lambda^s_{jk}F^i_{sh} - (k/h)) + \overset{\circ}{\Lambda}{}^s_{js}R^s_{kh}, \tag{4.9}$$

where

$$\Lambda_j{}^i{}_{kh} = \delta_h\Lambda^i_{jk} + \Lambda^s_{jk}\Lambda^i_{sh} - (k/h). \tag{4.9$'$}$$

5 Parallel and concurrent Finsler vector fields

Let $B^i(x,y)$ be a Finsler vector field and $F\Gamma(N)$ be a Finsler connection. Then it is said that (B^i) is parallel if

$$B^i_{|k} = 0,\ B^i\big|_k = 0 \tag{5.1}$$

and (B^i) is concurrent if

$$B^i_{|k} = -\delta^i_k,\ B^i\big|_k = 0. \tag{5.2}$$

It is our purpose to confirm the correctness of these definitions from the viewpoint of the almost Kählerian model of a Finsler space (see [9], ch.VII for details on this model). A different confirmation of these definitions is given in [8] using the principal Finsler bundle model due to M. Matsumoto. The giving of N is equivalent to the decomposition

$$T_uTM = H_uTM \oplus V_uTM,\ u \in TM \text{ (Whitney' sum)}. \tag{5.3}$$

Accordingly we have two projectors h and v and an almost product structure P such that if we put $X = hX + vX$ for a vector field X on TM, then

$$P(hX) = hX,\ P(vX) = -vX. \tag{5.5}$$

The set of Finsler connections is in a one–to–one correspondence with the set of linear connections on TM which verify

$$D_XP = 0,\ D_XF = 0 \text{ for any vector field } X \text{ on } TM. \tag{5.6}$$

By the very definition, a vector field B on TM is parallel with respect to D if

$$D_XB = 0, \tag{5.7}$$

and is concurrent if

$$D_XB = -X, \text{ for any vector field } X \text{ on } TM. \tag{5.8}$$

Let $(\delta_i, \dot{\partial}_i)$ be the usual adapted basis for the decomposition (5.3). The above mentioned one–to–one correspondence is established by

$$\begin{aligned} D_{\delta_k}\delta_j &= L^i_{jk}\delta_i, \quad D_{\dot{\partial}_k}\delta_j = V^i_{jk}\delta_i,\\ D_{\delta_k}\dot{\partial}_j &= L^i_{jk}\dot{\partial}_i, \quad D_{\dot{\partial}_k}\dot{\partial}_j = V^i_{jk}\dot{\partial}_i, \end{aligned} \tag{5.9}$$

for $D \leftrightarrow F\Gamma(N) = (N^i_j, L^i_{jk}, V^i_{jk})$.

It is obvious that (5.7) is equivalent to

$$D_{\delta_k}B = 0,\ D_{\dot{\partial}_k}B = 0, \tag*{(5.7)$'$}$$

and (5.8) is equivalent to

$$D_{\delta_k}B = -\delta_k,\ D_{\dot{\partial}_k}B = -\dot{\partial}_k. \tag*{(5.8)$'$}$$

Let now be $B = B^i(x,y)\delta_i + \hat{B}^i(x,y)\dot{\partial}_i$. Then $(5.7)'$ is equivalent by virtue of (5.9) with

$$(5.7)'' \qquad B^i_{|k} = 0,\ B^i\big|_k = 0,\ \hat{B}^i_{|k} = 0,\ \hat{B}^i\big|_k = 0.$$

One may associate to $B^i(x,y)$ at least the following three vector fields on TM : $B^i\delta_i$, $B^i\dot{\partial}_i$, $B^i\delta_i + B^i\dot{\partial}_i$ and it is obvious by $(5.7)''$ that $B^i(x,y)$ is parallel in the sense of (5.1) if and only if at least one from these vector fields on TM is parallel with respect to D. Thus (5.1) is in agreement with the usual definition of parallelism.

Let us make a similar analysis for concurrent Finsler vector fields. By (5.8), B is concurrent on TM if and only if

$$(5.10) \qquad B^i_{|k} = -\delta^i_k,\ B^i\Big|_k = 0,\ \hat{B}^i_{|k} = 0,\ \tilde{B}^i\big|_k = -\delta^i_k.$$

Now we assume that D or $F\Gamma(N)$ is of Cartan type, that is,

$$(5.11) \qquad y^i_{|k} = 0,\ y^i\big|_k = \delta^i_k.$$

The tensors $y^i_{|k}$ and $y^i\big|_k$ are called h–deflection and v–deflection tensors, respectively. The equations (5.11) hold for all four remarkable connections in Finsler spaces.

If moreover we assume that $\hat{B}^i$ is positively homogeneous of degree 1, a natural assumption in Finslerian setting, writing $\hat{B}^i\big|_k = -\delta^i_k$ in the form $\dot{\partial}_k\hat{B}^i + V^i_{jk}\hat{B}^j = -\delta^i_k$ and contracting it by y^k it results using (5.11) that $y^k\dot{\partial}_k\hat{B}^i = -y^i$. Thus by the Euler theorem, $\hat{B}^i = -y^i$ and then $\hat{B}^i_{|k} = 0$ reduces to $y^i_{|k} = 0$ i.e. the first equation in (5.11). Concluding, if we associate to the Finsler vector field $B^i(x,y)$ the vector field $B = B^i(x,y)\delta_i - y^i\dot{\partial}_i$ on TM, we find that $(B^i(x,y))$ is concurrent in the sense of (5.2) if and only if B is concurrent by the new definition of concurrence on any manifold. In other words, the condition (5.2) is in agreement with the notion of concurrence for vector fields.

6 The metric $^*g_{ij}$ with $B^i(x,y)$ a concurrent Finsler vector field

In this section we are dealing with the GL–metric $^*g_{ij}$ given by (3.1) for $B^i(x,y)$ a concurrent Finsler vector field with respect to the Cartan connection $C\Gamma$ of F^n i.e.

$$(6.1) \qquad B^i_{|j} = -\delta^i_j,\ B^i\big|_j = 0.$$

First we notice some results on concurrent Finsler vector fields due to M. Matsumoto and K. Eguchi [8].

If $B^i(x,y)$ is concurrent we have with respect to $C\Gamma$:

(6.2) $B_{i|j} = -g_{ij},\ B_i\big|_j = 0,$

(6.3) $B^h R_{hijk} = 0,\ B^h P_{hijk} + C_{ijk} = 0,\ B^h S_{hijk} = 0,$

(6.4) $B^i C_{ijk} = C^s_{jk} B_s = 0,$

(6.5) $B^i = B^i(x)$ and $B_i = B_i(x)$ i.e. B^i and B_i are functions on position only,

(6.6) $\partial_i B_j = \partial_j B_i = F^s_{ij} B_s - g_{ij},\ \partial_k B^i = -\delta^i_k - F^i_{sk} B^k.$

In these circumstancies a direct calculation yields

$$
\begin{aligned}
\Lambda^i_{jk} &= \frac{^*\sigma}{2\sigma} B^i(\sigma_k B_j + \sigma_j B_k + \sigma(B^s\sigma_s)B_j B_k - 2\sigma g_{jk}) - \frac{1}{2}\sigma^i B_j B_k \\
\overset{\circ}{\Lambda}{}^i_{jk} &= \frac{^*\sigma}{2\sigma} B^i(\dot\sigma_k B_j + \dot\sigma_j B_k + \sigma(B^s\dot\sigma_s)B_j B_k - \frac{1}{2}\dot\sigma^i B_j B_k, \text{ where}
\end{aligned}
\tag{6.7}
$$

$$\sigma_k := \delta_k\sigma,\ \dot\sigma_k := \dot\partial_k\sigma,\ \sigma^i = g^{ik}\sigma_k,\ \dot\sigma^i = g^{ik}\dot\sigma_k. \tag{6.7$'$}$$

Looking at (6.7) we see that the simplest case is given by

$$\sigma_k = 0,\ \dot\sigma_k = 0. \tag{6.8}$$

From (6.8) it results that σ is a constant c. And $^*F^2 := {}^*g_{ij}y^i y^j$ takes the form

$$^*F^2 = F^2 + c\beta^2,\ \beta = B_i(x)y^i. \tag{6.9}$$

Thus, for $c > 0$, $^*F$ is a new Finsler function which is obtained from F by a particular β–change.

The case $c = 1$ is studied in [8].

Further on we have

$$^*F^i_{jk} = F^i_{jk} - {}^*\sigma B^i g_{jk},\ {}^*C^i_{jk} = C^i_{jk}. \tag{6.10}$$

Remark 6.1. The Cartan nonlinear connection of $^*F^n = (M, {}^*F)$ is given by $N^i_j = \overset{c}{N}{}^i_j - \overset{*}{\sigma}\, B^i y_j$ i.e. the difference tensor is $A^i_j = \overset{*}{\sigma}\, B^i y_j$. Inserting it in (4.3)$'$ we find $^sF^i_{jk} = {}^*F^i_{jk}$. Therefore, in the geometry of $^*F^n$ we may equally use $\overset{c}{N}{}^i_j$ or N^i_j.

By (6.10) we immediately get

$$^*S_{ijkh} = S_{ijkh}. \tag{6.11}$$

Again by (6.10) but after a long calculation one finds

$$^*R_{ijkh} = R_{ijkh} + {}^*\sigma(g_{ik}g_{jh} - g_{ih}g_{jk}). \tag{6.12}$$

This suggests us to take into consideration the case when F^n is h–isotropic i.e. there exists a constant K such that $R_{ijkh} = K(g_{ik}g_{jh} - g_{ih}g_{jk})$. A contraction

of this last equation by B^i gives for $K \neq 0$, $B_k g_{jh} - B_h g_{jk} = 0$ in virtue of (6.3). A new contraction by B^k yields $B^2 g_{jh} = B_j B_h$ which contradicts the condition rank $(g_{ij}) = n > 1$. Thus we have

Theorem 6.1. *If F^n is h–isotropic, then it does not admit any concurrent Finsler vector field.*

The proof of the following two theorems are the same as for $c = 1$ (see Theorems 14 and 15 in [8]).

Theorem 6.2. *If F^n admits a concurrent Finsler vector field, then there is no a Finsler vector field which to be concurrent with respect to ${}^*F$ given by* (6.9).

Theorem 6.3. *If F^n admits a concurrent Finsler vector field and is R3–like, then ${}^*F^n = (M, {}^*F)$ with ${}^*F$ from* (6.5) *is also R3–like.*

Now we consider a more complicated case

$$\sigma_k = 0,\ \dot{\sigma}_k \neq 0. \tag{6.13}$$

Remark 6.2. The equation $\sigma_k := \dfrac{\partial \sigma}{\partial x^h} - \overset{c}{N}{}^s_k \dfrac{\partial \sigma}{\partial y^s} = 0$ is known as Tavakol–Van der Berg equation. A solution of it is for instance $\sigma = aF^2$ for $a \in \mathbb{R}$. For more details see [12].

Now (6.10) is replaced by

$$\begin{aligned} &{}^*F^i_{jk} = F^i_{jk} - {}^*\sigma B^i g_{jk} \\ &{}^*C^i_{jk} = C^i_{jk} + \frac{{}^*\sigma}{2\sigma} B^i(\dot{\sigma}_k B_j + \dot{\sigma}_j B_k + \sigma(B^s \dot{\sigma}_s) B_j B_k) - \frac{1}{2}\dot{\sigma}^i B_j B_k. \end{aligned} \tag{6.14}$$

The Remark 6.1 is still valid for this case. Precisely, if we ask for the vanishing of the h–deflection of ${}^*F\Gamma(\overset{c}{N})$, then ${}^*N^i_j = \overset{c}{N}{}^i_j - \overset{*}{\sigma}\ B^i y_j$ and so ${}^sF\Gamma(\overset{*}{N})$ coincides with ${}^*F\Gamma(\overset{c}{N})$.

Now we notice

$$ {}^*C_j = C_j + \frac{{}^*\sigma B^2}{2\sigma}\dot{\sigma}_j,\ C_j := C^i_{ji}, \tag{6.15}$$

$$ {}^*C_{jik} = C_{jik} + \frac{1}{2}(\dot{\sigma}_k B_i B_j + \dot{\sigma}_j B_i B_k - \dot{\sigma}_i B_j B_k). \tag{6.16}$$

A long calculation yields

$$\begin{aligned} &{}^*R_{jskh} = R_{jskh} + {}^*\sigma(g_{jk}g_{sh} - g_{jh}g_{sk}) + \\ &+ \frac{{}^*\sigma}{\sigma} B_s(\partial_k \sigma \cdot g_{jh} - \partial_h \sigma \cdot g_{jk}) + \frac{1}{2} B_j B_s R^q_{kh}\dot{\sigma}_q. \end{aligned} \tag{6.17}$$

Let us assume that F^n is a locally Minkowski space. Then $R_j{}^i{}_{kh} = 0$ and $C^i_{jk|h} = 0$. In a local chart in which g_{ij} do not depend on x we have $\overset{c}{N}{}^i_j = 0$ and so $\partial_k \sigma = \overset{c}{N}{}^p_j \dot{\sigma}_p = 0$ i.e. σ does not depend on x.

The equation (6.17) reduces to

$$
{}^*R_{jskh} = {}^*\sigma(g_{jk}g_{sh} - g_{jh}g_{sk}). \tag{6.18}
$$

It takes also the form

$$
\begin{aligned}
&{}^*R_{jskh} = {}^*\sigma({}^*g_{jk}{}^*g_{sh} - {}^*g_{jh}{}^*g_{sk}) + \sigma^*\sigma(B_jB_{hsk} + B_sB_{kjh}) \text{ for}\\
&B_{hsk} := B_hg_{sk} - B_kg_{sh}.
\end{aligned} \tag{6.18$'$}
$$

We notice that B_{hsk} is never vanishing since otherwise a contraction by B^h gives a contradiction with rank $(g_{ij}) = n > 1$.

7 A Beil metric for a Finsler space with (α, β)–metric

Here we consider again the Beil metric described in Remark 3.5. Let F^n be a Finsler space with an (α, β)–metric. A natural Beil metric is then

$$
{}^*g_{ij}(x, y) = a_{ij}(x) + \sigma(x, y)b_i(x)b_j(x). \tag{7.1}
$$

Let γ^i_{jk} be the Christoffel symbols for $a_{ij}(x)$. Then $\overset{c}{N}{}^i_j = \gamma^i_{jk}y^k =: \gamma^i_{j0}$ and the triple $\Gamma = (\gamma^i_{j0}, \gamma^i_{jk}, 0)$ may be thought of as a Finsler connection.

We have

Theorem 7.1. *If $b_i(x)$ is parallel and σ is covariant constant with respect to Γ, then Γ is like Chern–Rund connection for $({}^*g_{ij})$.*

Proof. Let $_{;k}$ denote the h–covariant derivative with respect to Γ. Notice that v–covariant derivative is just the derivative with respect to y. Our hypothesis read

$$
b_{i;k} = 0,\ \delta_k\sigma = 0,\ \delta_k = \partial_k - \gamma^s_{k0}\dot{\partial}_s. \tag{7.2}
$$

Then we easily get

$$
\begin{aligned}
{}^*g_{i;jk} &= (\delta_k\sigma)b_ib_j = 0\\
{}^*g_{ij,k} &= (\dot{\partial}_k\sigma)b_ib_j = 2{}^*C_{ikj}.
\end{aligned} \tag{7.3}
$$

Thus Γ is h–metrical and no metrical for ${}^*g_{ij}$. Hence it is similar to the Chern–Rund connection from Finsler geometry. **q.e.d.**

The Chern–Rund connection is a remarkable one in Finsler geometry ([1]). Notice that its h–deflection vanishes.

From now on we assume $b_{i;k} = 0$ and $\delta_k\sigma = 0$.

A direct calculation yields

$$
\begin{aligned}
{}^*F^i_{jk} &= \gamma^i_{jk},\\
{}^*C^i_{jk} &= \frac{{}^*\sigma}{2\sigma}b^i(\dot{\sigma}_kb_j + \dot{\sigma}_jb_k + \sigma(b^h\dot{\sigma}_h)b_jb_k) - \frac{1}{2}\dot{\sigma}^ib_jb_k.
\end{aligned} \tag{7.4}
$$

The first equation in (7.4) is important in many respects. For instance using it we find the h–curvature of ${}^*F\Gamma(\overset{c}{N})$ in the form

$$(7.5)\qquad {}^*R_h{}^i{}_{jh} = \gamma_h{}^i{}_{jh} + \overset{\circ}{\Lambda}_h{}^i{}_s R^s_{jk},$$

where $\gamma_h{}^i{}_{jh}$ is the curvature tensor of $a_{ij}(x)$ and $R^i_{jk} = \gamma_0{}^i{}_{jk}$. Here, as before, the index 0 indicates the contraction by y. Consequently, (7.5) takes the form

$$(7.6)\qquad {}^*R_h{}^i{}_{jk} = (\delta^i_s\delta^r_h + \overset{\circ}{\Lambda}{}^i{}_{hs}y^r)\gamma_r{}^s{}_{jk}.$$

From Ricci identities we find $\gamma_i{}^s{}_{jk}b_s = 0$ and from (7.5) we deduce

$$(7.7)\qquad {}^*R_{hijk} = \gamma_{hijk} + \frac{1}{2}b_h b_i \gamma_0{}^s{}_{jk}\dot{\sigma}_s.$$

As for Ricci curvatures one finds

$$(7.8)\qquad {}^*R_{ij} = r_{ij},$$

where r_{ij} is the Ricci curvature for $(a_{ij}(x))$. From here it results

$$(7.9)\qquad {}^*R = r,$$

where ${}^*R$ and r are the scalar curvatures for $({}^*g_{ij})$ and $(a_{ij}(x))$, respectively.

So, the h–Einstein tensor field of ${}^*g_{ij}$ i.e. ${}^*E_{ij} ={}^* R_{ij} - \frac{1}{2}{}^*R^*g_{ij}$ is related to the Einstein tensor E_{ij} of $a_{ij}(x)$ by

$$(7.10)\qquad {}^*E_{ij} = E_{ij} + \frac{\sigma r}{2}b_i b_j.$$

Consequently, the h–Einstein equation for GL i.e. ${}^*E_{ij} = \kappa^*\tau_{ij}$ with $\kappa \in \mathbb{R}$ reduces to

$$(7.11)\qquad r_{ij} - \frac{r}{2}a_{ij} = \kappa\tau_{ij},$$

where

$$(7.12)\qquad \tau_{ij} = {}^*\tau_{ij} - \frac{\sigma r}{2\kappa}b_i b_j.$$

The equation (7.11) is the Einstein equation for $(M, a_{ij}(x))$ but with the energy–momentum tensor influenced by a field described by b_i. In the the unified theory of R.G. Beil the term $b_i b_j$ in (7.12) is a "matter term" which could be the energy density of the self-field of a charged object.

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University "Al.I.Cuza" Iaşi
Faculty of mathematics
6600, Iaşi, Romania

Hokkaido Tokai University
Minami-ku, Minami-sawa
5-1, Sapporo 005, Japan

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LOCALLY CONFORMAL KÄHLER STRUCTURES ON TANGENT MANIFOLD OF A SPACE FORM

by Mihai ANASTASIEI

Abstract

A set of locally conformal Kähler structures on tangent manifold TM of a space form M is pointed out. This is found in a study of a type of Sasaki metric whose second term is a special deformation of the first one. Introducing an adequate almost complex structure we find at first a large class of locally conformal almost Kähler structures on TM for M a (pseudo)- Riemannian manifold. When M is a space form, a subset of it is made of locally conformal Kähler structures. One of them was found by R. Miron in [3].

1 Introduction

Let (M, g) be a (pseudo)-Riemannian manifold and $\triangledown$ its Levi-Civita connection. In a local chart $(U, (x^i))$ we set $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_i : \dfrac{\partial}{\partial x^i}$ and we denote by $\gamma^i_{jk}(x)$ the Christoffel symbols giving ∇. Let $(x^i, y^i) \equiv (x, y)$ be the local coordinates on the manifold TM projected on M by τ. The indices $i, j, k...$ will run from 1 to $n = \dim M$.

The functions $N^i_j(x, y) := \gamma^i_{jk}(x)y^k$ are the local coefficients of a nonlinear connection, that is the local vector fields $\delta_i = \partial_i - N^k_i(x, y)\, \dot{\partial}_k$, where $\dot{\partial}_k$: $\dfrac{\partial}{\partial y^k}$ span a distribution on TM called horizontal which is supplementary to the vertical distribution $u \to V_uTM = \ker \tau_{*,u}, u \in TM$. Let us denote by $u \to H_uTM$ the horizontal distribution and let $(\delta_i, \dot{\partial}_i)$ be the basis adapted to the decomposition $T_uTM = H_uTM \oplus V_uTM, u \in TM$. The basis dual of it is $(dx^i, \delta y^i)$ with $\delta y^i = dy^i + N^i_k(x, y)dx^k$.

The Sasaki metric on TM is as follows

$$G_S = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j. \tag{1.1}$$

If in the second term of G_S one replaces $g_{ij}(x)$ with the components $h_{ij}(x, y)$ of a generalized Lagrange metric (see Ch. X in [4]) one gets a type of Sasaki metric

$$G(x, y) = g_{ij}(x)dx^i \otimes dx^j + h_{ij}(x, y)\delta y^i \otimes \delta y^j. \tag{1.2}$$

In particular, $h_{ij}(x,y)$ could be a deformation of $g_{ij}(x)$, a case studied by the present author and H. Shimada in [1].

In this paper we are concerning with the metrical structure (1.2) in the case when $h_{ij}(x,y)$ is the following special deformation of $g_{ij}(x)$

$$h_{ij}(x,y) = a(L^2)g_{ij}(x) + b(L^2)y_iy_j, \tag{1.3}$$

where $L^2 = g_{ij}(x)y^iy^j, y_i = g_{ij}(x)y^j$ and $a, b : \mathrm{Im}(L^2) \subseteq \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $a > 0, b \geq 0$.

For $b = 0$ and $a = \dfrac{c^2}{L^2}$ for any constant c, the metrical structure (1.2), (1.3) was studied by R. Miron in [3] as an homogeneous lift of $g_{ij}(x)$ to TM.

In the following Section we introduce an almost complex structure which paired with G given by (1.2), (1.3) provides a large set of almost Hermitian structures on TM. Then, in Section 3 we show that all these structures are locally conformal almost Kähler structures. Finally, we find in Section 4 that, when (M,g) is of constant curvature, a part of them are locally conformal Kähler structures.

2 Some almost Hermitian structures on TM

Let F_S be the almost complex structure on TM given in the adapted basis $(\delta_i, \dot{\partial}_i)$ by

$$F_S(\delta_i) = -\dot{\partial}_i, F_S(\dot{\partial}_i) = \delta_i. \tag{2.1}$$

It is well known that the pair $(G_S,\ F_S)$ is an almost Kähler structure on TM, that is $G_S(F_SX, F_SY) = G_S(X,Y)$ and the 2-form

$$\Omega(X,Y) = G_S(F_S(X),Y) \text{ is closed, } X,Y \in \chi(M).$$

The pair (G, F_S) with G given by (1.2), (1.3) is no longer an almost Hermitian structure. We look for a new almost complex structure which paired with G to provide an almost Hermitian structure. We modify F_S to a linear map F given in the basis $(\delta_i, \dot{\partial}_i)$ as follows

$$F(\delta_i) = (\alpha\delta_i^k + \beta y_iy^k)\,\dot{\partial}_k, F(\dot{\partial}_j) = (\gamma\delta_j^h + \delta y_jy^h)\delta_h, \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta$ are functions on TM to be determined. The condition $F^2 = -I$ (identity) leads to

$$\alpha\gamma = -1, \alpha\delta + \beta\gamma + \beta\delta L^2 = 0. \tag{2.3}$$

Then the condition $G(F(X),F(Y)) = G(X,Y)$ gives

$$a\alpha^2 = 1, \gamma^2 = a, 2\gamma\delta + \delta^2L^2 = b, (2\alpha\beta + \beta^2L^2)(a + bL^2) + b\alpha^2 = 0 \tag{2.4}$$

The solution of the system of equations (2.3), (2.4) is

$$\alpha = -\frac{1}{\sqrt{a}}, \beta = \frac{\sqrt{a}+\sqrt{a+bL^2}}{L^2\sqrt{a(a+bL^2)}}, \gamma = \sqrt{a}, \delta = -\frac{\sqrt{a}+\sqrt{a+bL^2}}{L^2}. \tag{2.5}$$

We notice that for $b=0$, besides the solution provided by (2.5), that is

$$\alpha = -\frac{1}{\sqrt{a}}, \gamma = \sqrt{a}, \beta = \frac{2}{L^2\sqrt{a}}, \delta = -\frac{2\sqrt{a}}{L^2}, \tag{2.6}$$

there exists also the solution

$$\alpha = -\frac{1}{\sqrt{a}}, \gamma = \sqrt{a}, \beta = 0, \delta = 0. \tag{2.7}$$

Let us make the substitution $a \longrightarrow \dfrac{a^2}{L^2}$, $b \longrightarrow \dfrac{b^2-a^2}{L^4}$. Then (2.5) and (2.6) are unified to

$$\alpha = -\frac{L}{a}, \beta = \frac{a+b}{abL}, \gamma = \frac{a}{L}, \delta = -\frac{a+b}{L^3}, b \geq a > 0 \tag{2.8}$$

and (2.7) modifies to

$$\alpha = -\frac{L}{a}, \gamma = \frac{a}{L}, \beta = \delta = 0. \tag{2.9}$$

The metric G takes the form

$$G_{a,b}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \left(\frac{a^2}{L^2}g_{ij}(x) + \frac{b^2-a^2}{L^4}y_iy_j\right)\delta y^i \otimes \delta y^j, \quad b \geq a > 0. \tag{2.10}$$

Let $F_{a,b}$ be the almost complex structures given by (2.2), (2.8) and F_a those given by (2.2), (2.9). Then the pairs $(G_{a,b}, F_{a,b})$ and $(G_{a,a}, F_a)$ are almost Hermitian structures on TM.

For $a^2 = \dfrac{L^2}{1+L^2}, b = L^2$, the metric $G_{a,b}(x,y)$ is the Cheeger-Gromoll metric, [5],[6]

$$G_{CG}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{1+L^2}(g_{ij}(x) + y_iy_j)\delta y^i \otimes \delta y^j. \tag{2.11}$$

If $a^2 = \varphi' L^2, b^2 = L^2(\varphi' + 2\varphi'' L^2)$ for $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varphi'(t) \neq 0, t \in \mathrm{Im}(L^2)$, one obtains the Antonelli - Hrimiuc metrical structure, [2]

$$G_{AH}(x,y) = g_{ij}(x)dx^i \otimes dx^j + (\varphi' g_{ij}(x) + 2\varphi'' y_iy_j)\delta y^i \otimes \delta y^j. \tag{2.12}$$

3 Locally conformal almost Kähler structures on TM

Let $\Omega(X,Y) = G_{a,b}(F_{a,b}X, Y), X, Y \in \chi(TM)$ be the 2-form associated to the almost Hermitian structure $(G_{a,b}, F_{a,b})$.

Theorem 3.1 *The almost Hermitian structures $(G_{a,b}, F_{a,b})$ are locally conformal almost Kählerian structures, that is*

$$d\Omega = \Omega \wedge \theta, \theta = \frac{2a'L + b}{aL} dL. \tag{3.1}$$

Proof. We shall check (3.1) on the basis $(\delta_i, \dot{\partial}_i)$. If we rewrite (2.2) in the form

$$F(\delta_i) = A_i^k \, \dot{\partial}_k, F(\dot{\partial}_i) = B_j^h \delta_h, \tag{3.2}$$

we easily get

$$\Omega(\delta_i, \delta_j) = 0, \Omega(\delta_i, \dot{\partial}_j) = A_i^k h_{kj}, \Omega(\dot{\partial}_j, \delta_i) = B_j^k g_{ki}, \Omega\,(\dot{\partial}_i, \dot{\partial}_j,) = 0, \tag{3.3}$$

with $A_i^k h_{kj} + B_j^k g_{ki} = 0$.

Thus Ω is completely determined by

$$\Omega_{ij} := B_j^k g_{ki} = \gamma g_{ij} + \delta y_i y_j; \Omega = \Omega_{ij} \delta y^i \wedge dx^j. \tag{3.4}$$

Next we have the following essential components of $d\Omega$:
$d\Omega(\delta_i, \delta_j, \dot{\partial}_k) = \delta_j \Omega_{ik} - \gamma_{ki}^s \Omega_{sj} - \delta_i \Omega_{jk} - \gamma_{kj}^s \Omega_{si}$,
$d\Omega(\delta_i, \dot{\partial}_j, \dot{\partial}_k) = \dot{\partial}_j \, \Omega_{ik} - \dot{\partial}_k \, \Omega_{ij}$.

Now we consider the Berwald connection $(N_j^i = \gamma_{kj}^i(x) y^k, \ \gamma_{kj}^i(x), 0)$ on TM (see Ch.8 in [4]) and denote by $|k$ its $h-$covariant derivative. Thus because of $\Omega_{jk|i} = \delta_i \Omega_{jk} - \gamma_{ji}^s \Omega_{sk} - \gamma_{ki}^s \Omega_{js}$, we have $d\Omega(\delta_i, \delta_j \ ; \partial_k) = \Omega_{ki|j} - \Omega_{kj|i}$.

The following formulae are verified by a direct calculation.

$$g_{ij|k} = 0, y^j_{|k} = 0, y_{i|k} = 0, \delta_k L^2 = 0, \delta_k \psi(L^2) = 0, \tag{3.5}$$

$\dot{\partial}_k \, y_i = g_{ik}, \dot{\partial}_k \, L^2 = 2y_k, \dot{\partial}_k \, \psi(L^2) = 2y_k \psi'(L^2)$,
for any $\psi : \mathrm{Im}(L^2) \subseteq R_+ \longrightarrow R_+$.

Using (3.5) it immediately results $\Omega_{kj|i} = 0$ and so $d\Omega(\delta_i, \delta_j, \dot{\partial}_k) = 0$. Consequently, $d\Omega$ is completely determined by $d\Omega(\delta_i, \dot{\partial}_j, \dot{\partial}_k) = (\dot{\partial}_j \, \gamma) g_{ik} - (\dot{\partial}_k \, \gamma) g_{jk} + (\dot{\partial}_j \delta) y_k y_i - (\dot{\partial}_k \, \delta) y_j y_i + \delta(g_{ij} y_k - g_{ik} y_j)$.

Inserting here $\dot{\partial}_j \, \gamma, stackrel.\partial_j \delta$ with γ, δ from (2.8) one arrives to

$$d\Omega(\delta_i, \dot{\partial}_j, \dot{\partial}_k) = (2\gamma' - \delta)(g_{ik} y_j - g_{ij} y_k) = \frac{2a'L^2 + b}{L^3}(g_{ik} y_j - g_{ij} y_k). \tag{3.6}$$

Let be $\theta_0 = dL^2 = 2y_i\delta y^i$. Thus $\theta_0(\delta_i) = 0$ and $\theta_0\ (\dot{\partial}_j) = 2y_j$. Evaluating $\Omega \wedge \theta_0$ on the basis $(\delta_i, \dot{\partial}_i)$ one finds the essential component

$$\Omega \wedge \theta_0(\delta_i, \dot{\partial}_j, \dot{\partial}_k) = 2(\Omega_{ik}y_j - \Omega_{ij}y_k) = \frac{2a}{L}(g_{ik}y_j - g_{ij}y_k). \tag{3.7}$$

Comparing (3.6) with (3.7) one obtains $d\Omega = \dfrac{2a'L^2 + b}{2aL^2}\Omega \wedge \theta_0$ which is just (3.1).

Obviously θ is globally defined. Moreover, θ is closed. This fact can be directly verified using (3.5) or by differentiating (3.1).

Looking at (3.6) we notice that contracting $g_{ik}y_j - g_{ij}y_k = 0$ with g^{ik} one gets $(n-1)y_j = 0$ which is a contradiction. Thus we have

Theorem 3.2 *The almost Hermitian structures $(G_{a,b}, F_{a,b})$ are almost Kähler structures if and only if*

$$2a'L^2 + b = 0, \tag{3.8}$$

holds good.

We put $t = L^2$ and think (3.8) as a first order differential equation: $2ta'(t) + b(t) = 0$. Its general solutions is $a(t) = c - \dfrac{1}{2}\displaystyle\int \frac{b(t)}{t}dt$ for a constant c. Thus for various functions b we find a set of pairs (a, b) for which (3.8) holds. Choosing among these pairs those which verify $b \geq a > 0$ we find a set of almost Kähler structures on TM. For instance, if we take $b(t) = 2t$ it results $a(t) = c - t$ and $b \geq a > 0$ holds if $\dfrac{c}{3} \leq L^2(x, y) < c$, for $c > 0$. When $a = b$, the equation (3.8) has the general solution $a(t) = \dfrac{c}{\sqrt{t}}$. It follows

Corollary 3.1 *The almost Hermitian structures $(G_{a,a}, F_{a,a})$ are almost Kähler structures if and only if $a(L^2) = \dfrac{c}{\sqrt{L^2}}, c > 0$.*

The almost Hermitian structures $(G_{a,a}, F_a)$ have to be separately considered. Repeating for them the proof of Theorem 3.1 one obtains

Theorem 3.3 *The almost Hermitian structures $(G_{a,a}, F_a)$ are locally conformal almost Kähler structures, that is*

$$d\Omega = \Omega \wedge \theta, \theta = \frac{2a'L - a}{aL}dL. \tag{3.9}$$

The following corresponds to Theorem 3.2

Theorem 3.4 *The almost Hermitian structures $(G_{a,a}, F_a)$ are almost Kähler structures if and only if $a = c\sqrt{L^2}, c > 0$.*

Proof. The almost Kähler condition is now $2a'L^2 - a = 0$. Integrating the equation $2a't - a = 0$ one gets $a = c\sqrt{t}$.

Remark. For $a = c\sqrt{L^2}, c > 0, G_{a,a}$ is very close to G_S which is obtained for $c = 1$.

4 Locally conformal Kähler structures on TM

In order to find conditions that $(G_{a,b}, F_{a,b})$ be a locally conformal Kähler structure we have to put zero for the Nijenhuis tensor field of $F := F_{a,b}$,

$$(4.1)\qquad N_F = [FX, F] - F[FX, Y] - F[X, FY] - [X, Y], X, Y \in \chi(TM).$$

As the evaluation of N_F on the basis $(\delta_i, \dot{\partial}_i)$ is in general very complicated we confine ourselves to the structures $(G_{a,a}, F_a)$. In this case, the conditions

$$(4.2)\qquad N_F(\delta_i, \delta_j) = 0, N_F(\delta_i, \dot{\partial}_j) = 0, N_F\ (\dot{\partial}_j, \dot{\partial}_k) = 0,$$

are equivalent with six equations. Three of them are identities because of $\delta_i\alpha = \delta_i\gamma = 0$ and the other three are each one equivalent with

$$(4.3)\qquad R^k_{ij} = \frac{2a'L^2 - a}{a^3}(y_j\delta^k_i - y_i\delta^k_j),$$

where $R^k_{ij} = R^k_{sij}(x)y^s$ and R^k_{sij} is the curvature tensor of ∇.

By a contraction with g_{rk} the Eq. (4.3) reduces to

$$(4.4)\qquad R_{srij}(x)y^s = \frac{2a'L^2 - a}{a^3}(g_{js}g_{ri} - g_{is}g_{rj})y^s.$$

The Eq. (4.4) remember us the condition that (M, g) is of constant curvature (space form). It suggests us to look for functions a such that $\frac{2a'L^2 - a}{a^3} = k$, where k is a constant. For $t = L^2$, solving the Bernoulli equation $a' = \frac{1}{2t}a + \frac{k}{2t}a^3$ one gets $a(L^2) = \sqrt{\frac{L^2}{c - kL^2}}$ for $c - kL^2 > 0$, where c is a constant of integration. For these functions a, the Eq. (4.4) becomes

$$(4.5)\qquad R_{srij}(x)y^s = -k(g_{js}g_{ri} - g_{is}g_{rj})y^s,$$

which says that (M, g) is of constant curvature $-k$. Thus we have proved

Theorem 4.1 *If the (pseudo)-Riemannian manifold (M, g) is of constant curvature $k \in R$, for $a(L^2) = \sqrt{\frac{L^2}{c + kL^2}}$ with c a constant such that $c + kL^2 > 0$, the structures $(G_{a,a}, F_a)$ are locally conformal Kähler structures on TM.*

The explicit form of these structures is as follows:

$$(4.6)\qquad G_{a,a}(x, y) = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{c + kL^2}(g_{ij}(x))\delta y^i \otimes \delta y^j.$$

$$(4.7)\qquad F_a(\delta_i) = -\sqrt{c + kL^2}\ \dot{\partial}_i, F_a(\dot{\partial}_i) = \frac{1}{\sqrt{c + kL^2}}\delta_i,$$

The 1-form θ is

$$(4.8)\qquad \theta = \frac{kL}{c + kL^2}dL.$$

Corollary 4.1 *For $a(L^2) = c_0\sqrt{L^2}$, with c_0 a strict positive constant, the pairs $(G_{a,a}, F_a)$ are Kähler structures on TM if and only if (M, g) is flat.*

Proof. If (M, g) is flat, by the Theorem 4.1 for $a(L^2) = c_0\sqrt{L^2}, c_0 = \dfrac{1}{\sqrt{c}}$, the pair $(G_{a,a}, F_a)$ is a locally conformal Kähler structure and by the Theorem 3.4 this is almost Kähler. Thus $(G_{a,a}, F_a)$ is a Kähler structure on TM. Conversely, if the pair $(G_{a,a}, F_a)$ with $a(L^2) = c_0\sqrt{L^2}$ is a Kähler structure, the Eq. (4.3) gives $R^k_{ij} = 0$, equivalently $R_{srij}(x) = 0$, that is (M, g) is flat.

Looking at (4.6) and (4.7) we see that the structures $(G_{a,a}, F_a)$ from Corollary 4.1 are very close to (G_S, F_S) which is obtained for $c = 1$. Thus the Corollary 4.1 covers a well-known result: (G_S, F_S) is a Kählerian structure if and only if (M, g) is flat.

Finally, we notice that for $c = 0$ and $k \longrightarrow \dfrac{1}{k^2}$ in (4.6) and (4.7) one obtains the locally conformal Kähler structure found by R. Miron in [3].

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DEFORMATIONS OF FINSLER METRICS

by Mihai ANASTASIEI and Hideo SHIMADA

Abstract

Let $F^n = (M, F(x,y))$ be a Finsler space and $g_{ij}(x,y)$ its Finsler metric. We consider a deformation of $g_{ij}(x,y)$ of the form

$$ {}^*g_{ij}(x,y) = a(x,y)g_{ij}(x,y) + b(x,y)B_i(x,y)B_j(x,y), \tag{1.1}$$

with two Finsler scalars $a > 0$, $b \geq 0$ and $B_i(x,y)$ a Finsler co-vector. It follows that ${}^*g_{ij}$ is a generalized Lagrange metric in Miron'sense, briefly a GL–metric, see the monograph by R. Miron and M. Anastasiei [8]. The metric ${}^*g_{ij}$ unifies the Antonelli metrics, the Miron–Tavakol metrics, the Synge metrics (all treated in [8]) as well as the Antonelli–Hrimiuc ϕ-Lagrange metrics, [2], the Beil metrics, [4], and the vertical part of the Cheeger–Gromoll metric, [10]. We prove some general results on the geometry of the GL- space $(M,{}^* g_{ij}(x,y)$. Next, the Levi-Civita connection and the curvature of a Riemannian metric on the tangent manifold TM, induced by g_{ij} and ${}^*g_{ij}$ are determined. These are used for the study of a Riemannian submersion involving the Cheeger–Gromoll metric.

1 Deformations of Finsler metrics

Let $F^n = (M, F)$ be a Finsler space with a smooth i.e. C^∞ manifold M and $F : TM \to R$, $(x,y) \mapsto F(x,y)$. Here $x = (x^i)$ are coordinates on M and $(x,y) = (x^i, y^i)$ are coordinates on the tangent manifold TM projected on M by τ. The indices $i, j, k, ...$ will run from 1 to $n = \dim M$ and the Einstein convention on summation is implied. The geometrical objects on TM whose local components change like on M i.e. ignoring their dependence on y, will be called Finsler objects as in [7] or d–objects as in [8].

We set $\partial_i := \dfrac{\partial}{\partial x^i}$, $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}$ and notice that the vertical subspace of T_uTM i.e. $V_uTM = \text{Ker}\,(D\tau)_u$, $u \in TM$, where $D\tau$ means the differential of τ, is spanned by $(\dot{\partial}_i)$. The d–objects can be expressed using $(\dot{\partial}_i)$.

The Finsler metric $g_{ij}(x,y) = \dfrac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$ will be assumed positive definite. We have $F^2(x,y) = g_{ij}(x,y)y^iy^j$ and F^2 will be called the absolute energy of F^n. Assume that F^n is endowed with a $d-$vector field or a Finsler vector

field $B = B^i(x,y)\dot{\partial}_i$ and let $B_i(x,y)dx^i$ the Finsler 1-form with $B_i = g_{ik}B^k$. Set $B^2 = B_iB^i$ and consider the following deformation of $g_{ij}(x,y)$:

$$^*g_{ij}(x,y) = a(x,y)g_{ij}(x,y) + b(x,y)B_i(x,y)B_j(x,y), \tag{1.1}$$

with two Finsler scalars $a > 0$, $b \geq 0$. The metric $^*g_{ij}$ is no longer a Finsler metric but it is a positive definite generalized Lagrange metric in Miron'sense, briefly a GL–metric, see Ch. X in [8]. It is easy to check that $^*g^{jk} = \frac{1}{a}g^{jk} - cB^jB^k$ is the inverse of $^*g_{ij}$ for $c = \frac{b}{a(a+bB^2)}$.

Various particular forms of $^*g_{ij}(x,y)$ were previously considered by some authors. The conformal case i.e. $b = 0$, $a = exp(2\sigma(x,y))$ was studied and applied by R. Miron and R.K. Tavakol in *General Relativity*. The case $a = 1$ and $B_i = y_i$ provides, for a convenient form of $b(x,y)$, a metric which generalizes the Synge metric from Relativistic Optics. This case was studied by R. Miron and T. Kawaguchi. For $b = 0$, $a = exp(2\sigma(x))$ and $g_{ij}(x,y) = g_{ij}(y)$ one gets the Antonelli metric which was used in Ecology. For the results on all these metrics we refer to the chapters XI and XII in [8] and the references therein. The case $a = b = 1$ and $B_i(x,y) = B_i(x) = \frac{\partial f}{\partial x^i}$, $f : M \to \mathbb{R}$ was considered by C. Udrişte in [11] for studying the completeness of a Finsler manifold. The Riemannian version of this case i.e. $g_{ij}(x,y) = g_{ij}(x)$ was intensively used by Th. Aubin in [3]. The case $a = 1$ and $g_{ij}(x,y) = g_{ij}(x)$ with various choices of b and B_i was introduced and studied by R. G. Beil for constructing a new unified field theory, [5].

One says that $^*g_{ij}$ is reducible to a Lagrange metric, shortly an L–metric if there exists a Lagrangian $L : TM \to \mathbb{R}$ such that $^*g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_jL$. A necessary and sufficient condition for $^*g_{ij}$ be reducible to an L–metric is the symmetry in all indices of the Cartan tensor field $^*C_{ijk} = \frac{1}{2}\dot{\partial}_k{}^*g_{ij}$ i.e.

$$\dot{\partial}_k{}^*g_{ij} = \dot{\partial}_i{}^*g_{kj}. \tag{1.2}$$

Using (1.1) this condition becomes

$$\begin{gathered} \dot{a}_kg_{ij} - \dot{a}_jg_{ik} + \dot{b}_kB_iB_j - \dot{b}_jB_iB_k + b(\dot{\partial}_kB_i \cdot B_j - \dot{\partial}_iB_k \cdot B_j + \\ + B_i \cdot \dot{\partial}_kB_j - B_k \cdot \dot{\partial}_iB_j) = 0,\ \dot{a}_k := \dot{\partial}_ka, \dot{b}_k := \dot{\partial}_kb. \end{gathered} \tag{1.3}$$

Now we suppose that $a(x,y) = a(F^2)$ and $b(x,y) = b(F^2)$ assuming that the ranges of the real functions a and b from the right hand are included in $Im(F^2)$. It results $\dot{\partial}_ia = 2a'(F^2)y_i$ because of $\dot{\partial}_iF^2 = 2y_i$. Similarly, $\dot{\partial}_ib = 2b'(F^2)y_i$. We take $B_i = y_i$. For the GL-metric (1.1) subjected to the above conditions, (1.3) reduces to

$$(2a - b')(g_{ij}y_k - g_{ik}y_j) = 0. \tag{1.4}$$

Now if the equation $g_{ij}y_k - g_{ik}y_j = 0$ is multiplied by g^{ij} one gets $(n-1)y_k = 0$ which is a contradiction for $n \geq 1$. Thus we have

Theorem 1.1. *The GL-metric (1.1) with $B_i = y_i$, $a(x,y) = a(F^2)$, $b(x,y) = b(F^2)$ is an L-metric if and only if $2a = b'$.*

As always we may take $a = \phi'$, it comes out that the metric from Theorem 1.1 is essentially the ϕ-Lagrange metric of Antonelli– Hrimiuc, [2], i.e.

$$ {}^*g_{ij}(x.y) = a g_{ij}(x,y) + 2a' y_i y_j \tag{1.5} $$

The Cheeger-Gromoll metric is a Riemannian metric on TM of the form

$$ G_{CG} = g_{ij} dx^i \otimes dx^j + \frac{1}{1+F^2}(g_{ij}(x) + y_i y_j)\delta y^i \otimes \delta y^j, \tag{1.6} $$

for $\delta y^i = dy^i + \gamma^i_{jk} y^j dx^k$, where γ^i_{jk} are the Christoffel symbols of $g_{ij}(x)$. This suggests considering the following GL-metric of type (1.1) which generalizes the "vertical part" in (1.6):

$$ {}^*g_{ij} = \frac{1}{1+F^2}(g_{ij}(x) + y_i y_j), \tag{1.7} $$

which we call a CGL–metric.

Corolary 1.1. *The CGL-metric* (1.7) *is never reducible to a L-metric nor to a Finsler metric.*

2 Metrical connection of the GL–space $(M, {}^*g_{ij}(x,y))$

The geometry of ${}^*g_{ij}(x,y)$ is naturally connected with the geometry of F^n. It is our purpose to express the geometrical objects associated to ${}^*g_{ij}(x,y)$ using similar ones for F^n. If $\gamma^i_{jk}(x,y)$ are the generalized Christoffel symbols for $g_{ij}(x,y)$ and we put $\gamma^i_{00} := \gamma^i_{jk} y^j y^k$, then $\overset{\circ}{N}{}^i_j = \frac{1}{2}\dot{\partial}_j \gamma^i_{00}$ are the local coefficients of the Cartan nonlinear connection. The Cartan connection for F^n is $C\Gamma = (\overset{\circ}{N}{}^i_j, F^i_{jk}, C^i_{jk})$, where

$$ \begin{aligned} F^i_{jk} &= \frac{1}{2} g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}), \\ C^i_{jk} &= \frac{1}{2} g^{ih}(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}), \end{aligned} \tag{2.1} $$

for $\delta_j = \partial_j - \overset{\circ}{N}{}^k_j \dot{\partial}_k$.

This connection is h–metrical, i.e. $g_{ij\overset{\circ}{|}k} = 0$ and v–metrical, i.e. $g_{ij\overset{\circ}{\big|}k} = 0$.

Here $\overset{\circ}{|}_k$ and $\overset{\circ}{\big|}_k$ denote the h– and v–covariant derivatives with respect to $C\Gamma$. Moreover, two torsions of it vanish. We may consider a similar connection

for ${}^*g_{ij}(x,y)$. Indeed, let ${}^*C\Gamma = (\overset{\circ}{N}{}^i_j, {}^*F^i_{jk}, {}^*C^i_{jk})$ be the d–connection given by

$$
\begin{aligned}
{}^*F^i_{jk} &= \frac{1}{2}{}^*g^{ih}(\delta_j{}^*g_{hk} + \delta_k{}^*g_{jh} - \delta_h{}^*g_{jk}),\\
{}^*C^i_{jk} &= \frac{1}{2}{}^*g^{ih}(\dot\partial_j{}^*g_{hk} + \dot\partial_k{}^*g_{jh} - \dot\partial_h{}^*g_{jk}).
\end{aligned}
\tag{2.2}
$$

This d–connection is h–metrical i.e. ${}^*g_{ij\overset{*}{|}k} = 0$ and v–metrical i.e. ${}^*g_{ij\overset{*}{\Big|}k} = 0$ and the torsions ${}^*T^i_{jk} := {}^*F^i_{jk} - {}^*F^i_{kj} = 0$, ${}^*S^i_{jk} := {}^*C^i_{jk} - {}^*C^i_{kj} = 0$. Moreover, when $\overset{\circ}{N}{}^i_j(x,y)$ is fixed, ${}^*C\Gamma$ is the unique d–connection with these properties. It will be called the canonical metrical connection of ${}^*g_{ij}(x,y)$. Using (1.1) in (2.2), after some calculation one gets

Proposition 2.1. *The metrical connection ${}^*C\Gamma$ is given by*

$$
{}^*F^i_{jk} = F^i_{jk} + \Phi^i_{jk}, \quad {}^*C^i_{jk} = C^i_{jk} + \Lambda^i_{jk},
\tag{2.3}
$$

$$
\begin{aligned}
\Phi^i_{jk} = \frac{1}{2}{}^*g^{ih}[&a_j g_{hk} + a_k g_{jk} - a_h g_{jk} + \delta_j(bB_kB_h) + \\
&+\delta_k(bB_jB_h) - \delta_k(bB_jB_k)] - acB^iB^hF_{jhk}
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\Lambda^i_{jk} = \frac{1}{2}{}^*g^{ih}[&\dot a_j g_{hk} + \dot a g_{jh} - \dot a_h g_{jk} + \dot\partial_j(bB_kB_h) + \\
&+\dot\partial_k(bB_jB_h) - \dot\partial_h(bB_jB_k)] - acB^iB^hC_{ihk}
\end{aligned}
\tag{2.5}
$$

with the notations

$$
\begin{aligned}
a_k = \delta_k a,\ \ \dot a_k = \dot\partial_k a, \quad F_{jhk} &= \frac{1}{2}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}),\\
C_{jhk} &= \frac{1}{2}(\dot\partial_j g_{kh} + \dot\partial_k g_{jh} - \dot\partial_h g_{jk}).
\end{aligned}
\tag{2.6}
$$

Proposition 2.2. *The torsions of ${}^*C\Gamma$ are as follows:*

$$
\begin{aligned}
&{}^*T^i_{jk} = 0,\ {}^*R^i_{jk} = R^i_{jk} := \delta_k \overset{\circ}{N}{}^i_j - \delta_j \overset{\circ}{N}{}^i_k,\ {}^*S^i_{jk} = 0\\
&{}^*P^i_{jk} = P^i_{jk} - \Phi^i_{jk} \ \textit{where}\ P^i_{jk} = \dot\partial_k N^i_j - F^i_{jk}\ \textit{and}\ {}^*C^i_{jk}\ \textit{from}\ (2.3).
\end{aligned}
\tag{2.7}
$$

Proposition 2.3. *The curvatures of ${}^*C\Gamma$ are as follows:*

$$
{}^*S_j{}^i{}_{kh} = S_j{}^i{}_{kh} + \Lambda_j{}^i{}_{kh} + (C^s_{jk}\Lambda^i_{sh} + \Lambda^s_{jc}C^i_{sh} - k/h),
\tag{2.8}
$$

$$
\Lambda_j{}^i{}_{kh} = \dot\partial_h\Lambda^i_{jk} + \Lambda^s_{jk}\Lambda^i_{sh} - k/h,
\tag{2.8$'$}
$$

where $-k/h$ means the subtraction of the preceding terms with k replaced by h.

$$(2.9) \qquad {}^*R_j{}^i{}_{kh} = R_j{}^i{}_{kh} + \Phi_j{}^i{}_{kh} + (F^s_{jk}\Phi^i_{sh} + \Phi^s_{jk}F^i_{sh} - k/h) + \Lambda^i_{js}R^s_{kh},$$

$$(2.9)' \qquad \Phi_j{}^i{}_{kh} = \delta_h\Phi^i_{jk} + \Phi^s_{jk}\Phi^i_{sh} - k/h,$$

$$(2.10) \quad {}^*P_j{}^i{}_{kh} = P_j{}^i{}_{kh} + \Phi^i{}_{jk\overset{\circ}{|}h} - \Lambda^i{}_{jh\overset{\circ}{|}k} + \Lambda^i_{js}P^s_{kh} + C^s_{kh}\Phi^i_{sj} + \Phi^s_{jk}\Lambda^i_{sh} - \Phi^i_{sk}\Lambda^s_{jh}.$$

3 On a Riemannian metric on TM

Let TM be the tangent manifold to M endowed with the fundamental Finsler function F and the Finsler metric $g_{ij}(x,y)$. Consider the Cartan nonlinear connection $(\overset{\circ}{N}{}^a_j(x,y))$ and then $(\delta_i = \partial_i - \overset{\circ}{N}{}^a_i\dot{\partial}_a, \dot{\partial}_a)$ is a local frame on TM adapted to the decomposition of T_uTM into a direct sum of vertical and horizontal subspaces. From now on we shall use two types of indices: $a, b, c, ...$ will indicate vertical components and $i, j, k, ...$ will indicate horizontal ones. All have the same range $\{1, 2, ..., n\}$. Let be $h_{ab}(x,y) = \delta^i_a\delta^j_b{}^*g_{ij}(x,y)$, where δ^i_a is the Kronecker symbol, and

$$(3.1) \qquad G(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b,$$

where $\delta y^a = dy^a + N^a_k(x,y)dx^k$.

Then $(TM, G(x,y))$ is an oriented Riemannian manifold. The horizontal and vertical distributions are mutually orthogonal with respect to G. It is our purpose to study the Riemannian metric G. First, we compute the coefficients of the Levi–Civita connection D of G in the frame $(\delta_i, \dot{\partial}_a)$. We set

$$(3.2) \qquad \begin{aligned} &D_{\delta_k}\delta_j = F^i_{jk}\delta_i + A^a_{jk}\dot{\partial}_a, \quad D_{\dot{\partial}_b}\delta_j = \widetilde{C}^i_{jb}\delta_i + E^a_{jb}\dot{\partial}_a, \\ &D_{\delta_k}\dot{\partial}_b = L^a_k\dot{\partial}_a + D^i_{bk}\delta_i, \quad D_{\dot{\partial}_b}\dot{\partial}_c = C^a_{cb}\dot{\partial}_a + B^i_{cb}\delta_i \end{aligned}$$

Let $\mathbb{T}$ be the torsion of D i.e. $\mathbb{T}(X,Y) = D_XY - D_YX - [X,Y]$ for X, Y vector fields on TM. The condition D is torsion–free is equivalent to

$$(3.3) \qquad \mathbb{T}(\delta_i, \delta_j) = \mathbb{T}(\delta_i, \dot{\partial}_a) = \mathbb{T}(\dot{\partial}_a, \dot{\partial}_b) = 0.$$

Using the following equations

$$(3.4) \qquad [\delta_i, \delta_j] = R^a_{ij}\dot{\partial}_a, \ [\delta_j, \dot{\partial}_b] = (\dot{\partial}_bN^a_j)\dot{\partial}_a, \ [\dot{\partial}_a, \dot{\partial}_b] = 0$$

where $R^a_{ij} = \delta_jN^a_i - \delta_iN^a_j$, one finds that (3.3) is equivalent to

$$(3.5) \qquad \begin{aligned} &F^k_{ij} = F^k_{ji}, \quad A^a_{ij} - A^a_{ji} = -R^a_{ij} \\ &D^k_{ai} = \widetilde{C}^k_{ia}, \quad L^b_{ai} = \dot{\partial}_aN^b_i + E^b_{ia} \\ &C^a_{bc} = C^a_{cb}, \quad B^i_{bc} = B^i_{cb}. \end{aligned}$$

The condition that D is metrical, that is, $XG(X,Y) = G(D_XY,Z)+G(Y,D_XZ)$, written in the frame $(\delta_i, \dot{\partial}_a)$ gives

$$
\begin{aligned}
&F^h_{ji}g_{hk} + F^h_{ki}g_{hj} = \delta_i g_{jk}, \quad \widetilde{C}^i_{ja}g_{ik} + \widetilde{C}^i_{ka}g_{ij} = \dot{\partial}_a g_{jk},\\
&A^c_{ji}h_{ca} + D^k_{ai}g_{kj} = 0, \qquad E^c_{ja}h_{cb} + B^k_{ba}g_{kj} = 0,\\
&L^c_{ai}h_{cb} + L^c_{bi}h_{ca} = \delta_i h_{ab}, \quad C^e_{ba}h_{ec} + C^e_{ca}h_{eb} = \dot{\partial}_a h_{bc}.
\end{aligned}
\tag{3.6}
$$

The systems (3.5) and (3.6) have the unique solution

$$
\begin{aligned}
&F^k_{ij} = \frac{1}{2}g^{kh}(\delta_i g_{hj} + \delta_j g_{hi} - \delta_h g_{ij}),\ A^a_{jk} = \frac{1}{2}(-R^a_{jk} - h^{ab}\dot{\partial}_b g_{jk}),\\
&\widetilde{C}^i_{jb} = \frac{1}{2}g^{ih}(\dot{\partial}_b g_{jh} + h_{bc}R^c_{hj)} = D^i_{bj},\\
&E^a_{ib} = \frac{1}{2}h^{ac}h_{bc\|i},\ L^a_{bi} = \dot{\partial}_b N^a_i + \frac{1}{2}h^{ac}h_{bc\|i},\\
&B^k_{ab} = -\frac{1}{2}g^{kj}h_{ab\|j},\ C^a_{bc} = \frac{1}{2}h^{ad}(\dot{\partial}_b h_{dc} + \dot{\partial}_c h_{bd} - \dot{\partial}_d h_{bc}).
\end{aligned}
\tag{3.7}
$$

Here $h_{bc\|i}$ denotes the h–covariant derivative of h_{bc} with respect to the Berwald connection $B\Gamma = (\overset{\circ}{N}{}^a_i, \dot{\partial}_b N^a_i, 0)$. Now we shall compute the components of the curvature of D in the same frame. To this aim we shall consider an intermediate linear connection ∇ on TM:

$$
\begin{aligned}
&\nabla_{\delta_j}\delta_k = F^i_{jk}\delta_i, \quad \nabla_{\dot{\partial}_b}\delta_j = D^i_{bj}\delta_i\\
&\nabla_{\delta_k}\dot{\partial}_b = L^a_{bk}\dot{\partial}_a, \quad \nabla_{\dot{\partial}_b}\dot{\partial}_c = C^a_{cb}\dot{\partial}_a.
\end{aligned}
\tag{3.8}
$$

This connection is metrical with respect to G i.e. $\nabla_X G = 0$, it preserves the horizontal and vertical distributions and it has three non-vanishing torsions: $R^a_{jk}, D^i_{bj},\ P^a_{jb} = \dfrac{1}{2}h^{ac}h_{bc\|j}$.

The curvature of ∇ has six components in the form (see p. 48 of [8]):

$$
\begin{aligned}
&\widehat{R_h{}^i{}_{jk}} = \delta_k F^i_{hj} + F^m_{hj}F^i_{mk} - j/k + D^i_{ah}R^a_{jk},\\
&\widetilde{R_b{}^a{}_{jk}} = \delta_k L^a_{bj} + L^c_{bj}L^a_{ck} - j/k + C^a_{bc}R^c_{jk},\\
&\widetilde{P_j{}^i{}_{ka}} = \dot{\partial}_a F^i_{jk} - D^i_{aj|k} + D^i_{bj}P^b_{ka},\\
&P_b{}^a{}_{kc} = \dot{\partial}_c L^a_{bk} - C^a_{bc|k} + C^a_{bd}P^d_{kc},\\
&\widetilde{S_j{}^i{}_{bc}} = \dot{\partial}_c D^i_{bj} + D^h_{bj}D^i_{ch} - b/c,\\
&S_b{}^a{}_{cd} = \dot{\partial}_d C^a_{bc} + C^e_{bc}C^a_{ed} - c/d.
\end{aligned}
\tag{3.9}
$$

Here and in the following $_{|k}$ and $\big|_a$ will denote h– and v–covariant derivatives with respect to ∇.

Remark 3.1. $S_b{}^a{}_{cd}$ is nothing but ${}^*S_j{}^i{}_{kh}$. And the other tensors in (3.9) can be expressed with $R_j{}^i{}_{kh}, P_j{}^i{}_{kh}, S_j{}^i{}_{kh}$ or with their $*$–counterparts. For instance, $\widehat{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} + \dfrac{1}{2}g^{is}h_{ac}R^c_{sh}R^a_{jk}$.

Let K be the curvature tensor field of the Levi–Civita connection D. We shall denote its components by the same letter K indexed with two types of indices with the understanding that different indices means different components. There will be twelve components of K. After calculation one finds

$$
\begin{aligned}
&K(\dot\partial_b,\dot\partial_c)\dot\partial_d := K_d{}^a{}_{cb}\dot\partial_a + K_d{}^i{}_{cb}\delta_i,\\
&K_d{}^a{}_{cb} = S_d{}^a{}_{cb} + B^i_{cd}E^a_{ib} - B^i_{db}E^a_{ic},\ K_d{}^i{}_{cb} = B^i_{\ cd\big|b} - B^i_{\ bd\big|c},\\
&K_{abdc} = S_{abdc} + \frac{1}{2}(B^i_{ad}h_{bc\|i} - B^i_{ac}h_{bd\|i},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&K(\dot\partial_b,\dot\partial_c)\delta_j = K_j{}^a{}_{cb}\dot\partial_a + K_j{}^i{}_{cb}\delta_i,\\
&K_j{}^i{}_{cb} = \widetilde{S_j{}^i{}_{cb}} + E^d_{jc}B^i_{db} - E^d_{jb}B^i_{dc},\ K_j{}^a{}_{cb} = E^a_{\ jc\big|b} - E^a_{\ jb\big|c},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&K(\dot\partial_b,\delta_j)\dot\partial_c := K_c{}^a{}_{jb}\dot\partial_a + K_c{}^i{}_{jb}\delta_i,\\
&K_c{}^a{}_{jb} = P_c{}^a{}_{jb} - B^k_{cb}A^a_{kj} + D^k_{cj}E^a_{kb},\\
&K_c{}^i{}_{jb} = D^i_{\ cj\big|b} - B^i_{bc|j} - P^d_{jb}B^i_{dc} + D^k_{bj}D^i_{ck},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&K(\dot\partial_b,\delta_k)\delta_j := K_j{}^a{}_{kb}\dot\partial_a + K_j{}^i{}_{kb}\delta_i,\\
&K_j{}^a{}_{kb} = A^a_{\ jk\big|b} - E^a_{jb|k} + D^h_{bk}A^a_{jh} + P^c_{kb}E^a_{jc},\\
&K_j{}^i{}_{kb} = \widetilde{P_j{}^i{}_{kb}} + A^c_{jk}B^i_{cb} - E^c_{jb}D^i_{ck},\\
&K_{jakb} = A_{\ ajk\big|b} - E_{ajb|k} + A_{ajh}D^h_{bk} + E_{ajc}P^c_{kb},
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&K(\delta_j,\delta_k)\dot\partial_b := K_b{}^a{}_{kj}\dot\partial_a + K_b{}^i{}_{kj}\delta_i,\\
&K_b{}^a{}_{kj} = \widetilde{R_b{}^a{}_{kj}} + D^h_{bk}A^a_{hj} - D^h_{bj}A^a_{hk},\\
&K_b{}^i{}_{kj} = D^i_{bk|j} - D^i_{bj|k} - R^c_{jk}B^i_{bc},
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&K(\delta_j,\delta_k)\delta_h := K_h{}^a{}_{kj}\dot\partial_a + K_h{}^i{}_{kj}\delta_i,\\
&K_h{}^i{}_{kj} = \widehat{R_h{}^i{}_{kj}} + A^b_{hk}D^i_{bj} - A^b_{hj}D^i_{bk},\\
&K_h{}^a{}_{kj} = A^a_{hk|j} - A^a_{hj|k} + R^c_{kj}E^a_{hc},\\
&K_{hikj} = R_{hikj} + D_{ibj}A^b_{hk} - D_{ibk}A^b_{hj}.
\end{aligned}
\tag{3.15}
$$

Now easily follows

Proposition 3.1. *The sectional curvatures of D are as follows:*

$$
\begin{aligned}
&K_{ab} = [S_{abab} + \frac{1}{2}(B^i_{aa}h_{bb\|i} - B^i_{ab}h_{ab\|i})]/(h_{aa}h_{bb} - h^2_{ab}),\\
(3.16)\qquad &K_{ja} = (A_{ajj}\big|_a - E_{aja|k} + A_{ajh}D^h_{aj} + E_{ajc}P^c_{ja})/g_{jj}g_{aa}\\
&K_{ji} = (R_{jiji} + D_{ibi}A^b_{jj} - D_{ibj}A^b_{ji})/(g_{ii}g_{jj} - g^2_{ij}).
\end{aligned}
$$

In the following we assume that F^n reduces to a Riemannian space i.e. $g_{ij}(x,y) = g_{ij}(x)$. The Cartan nonlinear connection reduces to $\overset{\circ}{N}{}^i_j(x,y) = \gamma^i_{jk}(x)y^k$, where $(\gamma^i_{jk}(x))$ are the Christoffel symbols of the metric $g = (g_{ij}(x))$. We consider the corresponding Riemannian metric G given by (3.1) and we have

Proposition 3.2. *The mapping $\tau : (TM, G) \to (M, g)$ is a Riemannian submersion.*

Indeed, τ is of maximal rank n and its differential $D\tau$ preserves the lengths of horizontal vectors as it follows from $G(\delta_i, \delta_j) = g_{ij}(x)$.

Let h and v denote the projections of T_uTM onto the subspaces of horizontal and vertical vectors, respectively. Following B. O'Neil, [9], the fundamental tensor fields of the Riemannian submersion τ are as follows:

$$(3.17)\qquad S(X,Y) = hD_{vX}Y + vD_{vX}hY,$$

$$(3.18)\qquad N(X,Y) = vD_{hX}hY + hD_{hX}vY, \quad X, Y \in \mathcal{X}(TM).$$

In the frame $(\delta_i, \dot{\partial}_a)$ we have

$$(3.19),\qquad S(\delta_i,\delta_j) = 0, S(\delta_i, \dot{\partial}_a) = 0, S(\dot{\partial}_a, \delta_i) = E^j_{ia}\delta_j, S(\dot{\partial}_a, \dot{\partial}_b) = B^i_{ab}\delta_i.$$

$$(3.20).\quad N(\delta_i,\delta_j) = \frac{1}{2}R^a_{ij}\dot{\partial}_a, N(\delta_i, \dot{\partial}_a) = D^i_{ai}\delta_j, N(\dot{\partial}_a, \delta_i) = 0, N(\dot{\partial}_a, \dot{\partial}_b) = 0$$

By (3.19) and (3.7) it follows

Proposition 3.3. *The Riemannian submersion $\tau : (TM, G) \to (M, g)$ is totally geodesics, i.e. $S = 0$ if and only if*

$$(3.21)\qquad {}^*g_{ij\|k} = 0,$$

where $_{\|k}$ denotes the h–covariant derivative with respect to the Berwald connection $(\gamma^i_{jk}(x)y^k, \gamma^i_{jk}(x), 0)$.

Proposition 3.4. *The tensor field N vanishes if and only if the Riemannian metric g is flat.*

4 Deformations of Riemannian metrics

The geometrical objects associated to ${}^*g_{ij}(x,y)$ are generally complicated. Some simplifications appear for particular choices of a, b and B_i. We studied in a previous paper, [1], the case $a = 1$ and a concurrent d-vector field

$B^i(x,y)$ while M. Kitayama studied the case $a=1$ and a parallel d-vector field $B^i(x,y)$, [6]. Here we selected for a detailed analysis the following deformation of a Riemannian metric $g=(g_{ij}(x))$:

$$
{}^*g_{ij}(x,y) = a(F^2)g_{ij}(x) + b(F^2)y_iy_j, \tag{4.1}
$$

where $F^2(x,y) = g_{ij}(x)y^iy^j, y_i = g_{ij}(x)y^j$.

Accordingly, we consider the Riemannian submersion $\tau : (TM, G) \to (M,g)$, where

$$
G(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + (a(F^2)g_{ij}(x) + b(F^2)y_iy_j)\delta y^a \otimes \delta y^b, \tag{4.2}
$$

The GL–metric (4.1) contains as a particular case the ϕ–Lagrange metric associated to a Riemannian space while G generalizes the Cheeger–Gromoll metric studied by Sekizawa [10]. The Cartan connection for $(M, g_{ij}(x))$ reduces to $C\Gamma = (\gamma^i_{jk}(x)y^j, \gamma^i_{jk}, 0)$. The v–covariant derivative $\overset{\circ}{\Big|}_k$ coincides with the partial derivative with respect to (y^k). The h–covariant derivative $\overset{\circ}{|}_k$ reduces to the usual covariant derivative for the objects which do not depend on (y^i) and coincides with $_{\|k}$ for the others.

We notice for the later use the following formulae

$$
\delta_k F^2 = 0,\ y^i_{\overset{\circ}{|}k} = 0,\ y_{i\overset{\circ}{|}k} = 0,\ y^i\overset{\circ}{\Big|}_k = \delta^i_k,\ y_i\overset{\circ}{\Big|}_k = g_{ik}(x) \tag{4.3}
$$

$$
\begin{aligned}
&\delta_k a = 0, && \delta_k b = 0\\
&\dot{\partial}_k a = 2a'y_k, && \dot{\partial}_k b = 2b'y_k.
\end{aligned} \tag{4.4}
$$

By a direct calculation one proves

Proposition 4.1. *The d–connection $C\Gamma$ of the GL–metric* (4.1) *is given by*

$$
\begin{aligned}
{}^*F^i_{jk} &= \gamma^i_{jk}(x) i.e. \quad \Phi^i_{jk} = 0\\
{}^*C^i_{jk}(x,y) &= \Lambda^i_{jk}(x,y) = \frac{a'}{a}(\delta^i_k y_j + \delta^i_j y_k) +\\
&+ \frac{b-a'}{a+bF^2}y^i g_{jk} + \frac{ab' - 2a'b}{a(a+bF^2)}y^i y_j y_k.
\end{aligned} \tag{4.5}
$$

From (4.3) and (4.4) it results

$$
{}^*g_{ij\overset{\circ}{|}k} = 0,\ {}^*g_{ij}\overset{\circ}{\Big|}_k = 2a'g_{ij}y_k + b(g_{ik}y_j + g_{jk}y_i) + 2b'y_iy_jy_k. \tag{4.6}
$$

Thus ${}^*g_{ij}$ is h–metrical and not v–metrical with respect to $C\Gamma$. The torsions of ${}^*C\Gamma$ of the GL–metric (4.1) are vanishing excepting ${}^*R^i_{jk} = \gamma^i_{hjk}(x)y^h$ and ${}^*C^i_{jk}$ from (4.5). As for its curvatures we find

$$
{}^*R_j{}^i{}_{kh} = r_j{}^i{}_{kh}(x) + \Lambda^i_{js}R^s_{kh}, \tag{4.7}
$$

$$(4.8)\qquad {}^*P_j{}^i{}_{kh} = 0 \text{ because of } \Lambda^i_{\substack{\circ\\ jh|k}} = 0,$$

$$(4.9)\qquad {}^*S_j{}^i{}_{kh} = \Lambda^i_{jkh} \text{ from } (2.8)',$$

where r^i_{jkh} is the curvature tensor of $(g_{ij}(x))$.

Using $y_s R^s_{kh} = y_s r_s{}^p{}_{kh} y^p = r_{pikh} y^p y^i = 0$, one gets

$$(4.7)'\qquad {}^*R_j{}^i{}_{kh} = r_j{}^i{}_{kh}(x) + \frac{a'}{a} y_j R^i_{kh} + \frac{b-a'}{a+bF^2} y^i R_{jkh},$$

$$(4.7)''\qquad {}^*R_0{}^i{}_{kh} = \left(1 + \frac{a'F^2}{a}\right) R^i_{kh},$$

where "0" denotes the contraction by (y^j).

Now we consider the Riemannian metric G given by (4.2). The Levi–Civita connection of it has the local coefficients

$$(4.10)\qquad \begin{aligned} &F^k_{ij} = \gamma^k_{ij}(x),\ A^a_{jk} = -\frac{1}{2} r_0{}^a{}_{jk},\\ &D^i_{bj} = \frac{a}{2} r_j{}^i{}_{b0} = \widetilde{C}^i_{jb},\\ &E^a_{ib} = 0 = B^k_{ab},\ L^a_{bi} = \gamma^a_{bi}(x),\ C^a_{bc} = \Lambda^a_{bc}. \end{aligned}$$

The curvature of ∇ from (3.9) reduces to

$$(4.11)\qquad \begin{aligned} &\widehat{R}_h{}^i{}_{jk} = r_h{}^i{}_{jk}(x) + \frac{a}{2} r_h{}^i{}_{a0} \cdot r_0{}^a{}_{jk},\\ &\widetilde{R_b{}^a{}_{jk}} = {}^*R_b{}^a{}_{jk},\\ &\widetilde{P_j{}^i{}_{ka}} = -\frac{a}{2} r_j{}^i{}_{a0;k},\\ &P_b{}^a{}_{kc} = 0 \text{ because of } \Lambda^a_{bc|k} = 0,\\ &\widetilde{S_j{}^i{}_{bc}} = a r_j{}^i{}_{bc} + \left(a' y_c r_j{}^i{}_{b0} + \frac{a^2}{4} r_j{}^h{}_{b0} r^i_{hc0} - b/c\right),\\ &S_b{}^a{}_{cd} = \Lambda^a_{bcd}. \end{aligned}$$

The curvature of the Levi–Civita connection D are given by

$$
(4.12)\qquad
\begin{aligned}
&K_d{}^a{}_{bc} = \Lambda_d{}^a{}_{bc},\ \ K_d{}^i{}_{cb} = 0,\\
&K_j{}^i{}_{bc} = \widetilde{S}_j{}^i{}_{bc},\ \ K_j{}^a{}_{bc} = 0,\\
&K_c{}^a{}_{jb} = 0,\ \ K_c{}^i{}_{jb} = \frac{a}{2} r_j{}^i{}_{cb} - \frac{a'}{2} y_b r_j{}^i{}_{c0} - \frac{a'}{2} y_c r_j{}^i{}_{b0} + \frac{a^2}{4} r_s{}^i{}_{b0} r_j{}^s{}_{c0},\\
&K_j{}^a{}_{kb} = -\frac{1}{2} r_b{}^a{}_{jk} + \frac{a}{4} r_0{}^a{}_{ik} - \frac{a'}{2a} y_b r_0{}^a{}_{jk} - \frac{b-a'}{2(a+bF^2)} y^a r_{0bjk},\\
&K_j{}^i{}_{kb} = -\frac{a}{2} r_j{}^i{}_{b0;k},\\
&K_b{}^a{}_{kj} = r_b{}^a{}_{kj} + \frac{a'}{a} y_b r_0{}^a{}_{kj} + \frac{b-a'}{a+bF^2} y^a r_{0bkj} - \frac{a}{4} r_k{}^h{}_{b0} r_0{}^a{}_{hk} +\\
&\qquad\qquad + \frac{a}{4} r_j{}^h{}_{b0} r_0{}^a{}_{jk},\\
&K_b{}^i{}_{kj} = \frac{a}{2}(r_k{}^i{}_{b0;j} - r_j{}^i{}_{b0;k}),\\
&K_h{}^i{}_{kj} = r_h{}^i{}_{kj} + \frac{a}{2} r_h{}^i{}_{a0} r_0{}^a{}_{kj} - \frac{a}{4} r_0{}^a{}_{hj} r_k{}^i{}_{a0},\\
&K_h{}^a{}_{kj} = \frac{1}{2}(r_0{}^a{}_{hj;k} - r_0{}^r{}_{ah} k; j).
\end{aligned}
$$

An inspection of (4.11) and (4.12) gives

Theorem 4.1. *If (M,g) is flat, then (TM,G) is flat if and only if $\Lambda_j{}^i{}_{kh} = 0$.*

This theorem shows that G is less "rigid" than the Sasaki metric of $(g_{ij}(x))$ which is locally flat if and only if $(g_{ij}(x))$ is locally flat.

Now if we fix $x = x_0$, then ${}^*g_{ij}(x_0,y)$ is a Riemannian metric in the fibre $T_{x_0}M$ and $\Lambda_j{}^i{}_{kh}$ is just its curvature tensor field. Thus we may reformulate Theorem **??** in the form

Theorem 4.1′. *If (M,g) is flat, then (TM,G) is flat if and only if $(T_{x_0}(M),\ {}^*g_{ij}(x_0,y))$ is a flat Riemannian manifold for every $x_0 \in M$.*

For the conformal case i.e. $b = 0$ one finds

$$
(4.13)\qquad \Lambda^i_{jk} = \frac{a'}{a}(\delta^i_k y_j - \delta^i_j y_k - y^i g_{jk})
$$

$$
(4.14)\qquad
\begin{aligned}
\Lambda_j{}^i{}_{kh} &= \left[2\left(\frac{a'}{a}\right)' - \frac{a'^2}{a}\right](\delta^i_k y_j y_h + y^i y_k g_{jh} - h/k) +\\
&+ \frac{a'^2}{a} F^2(\delta^i_k g_{jh} - \delta^i_h g_{jk}).
\end{aligned}
$$

It follows

Proposition 4.2. $\Lambda_j{}^i{}_{kh} = 0 \iff a = \text{constant}$.

From Theorem **??** and (4.6) one deduces

Proposition 4.3. *The Riemannian submersion* $\tau : (TM, G) \to (M, g)$ *with* G *given by* (4.2) *is totally geodesics.*

The other consequences of the previous formulae will be presented elsewhere.

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University "Al. I. Cuza" Iaşi
Faculty of Mathematics
6600, Iaşi, Romania

Hokkaido Tokai University
Minami-ku, Minami-sawa
5-1, Sapporo 005, Japan

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A FRAMED $f-$STRUCTURE ON TANGENT MANIFOLD OF A FINSLER SPACE

by Mihai ANASTASIEI

Abstract

It is shown that the slit tangent manifold $\overset{\circ}{TM}$ of a Finsler space $F^n = (M, L)$ carries a natural framed $f-$structure of corank 2. When this is restricted to the indicatrix bundle TM defined by $L = 1$, one gets a *contact Riemannian structure on* TM which is Sasakian iff F^n is of constant curvature 1.

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1 Introduction

Let M be a smooth i.e. C^∞ manifold of dimension n and $\tau : TM \to M$ its tangent bundle. If (x^i), $i, j, k... = 1, ..., n$ are local coordinates on M, the induced local coordinates on TM will be denoted by $(x, y) \equiv (x^i = x^i \circ \tau, y^i)$, where (y^i) are the components of a vector from T_pM, $p(x^i)$, in the natural basis $\left(\partial_i := \dfrac{\partial}{\partial x^i}\right)$.

Let $F^n = (M, L)$ be a Finsler space. The function $L :\overset{\circ}{TM}:= TM \setminus \{(x, 0)\} \to \mathbb{R}_+$ is smooth, positively homogeneous of degree 1 with respect to (y^i) and the matrix with the entries $g_{ij}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 L^2}{\partial y^i \partial y^j}$ is of rank n. From the homogeneity of L it follows $L^2(x, y) = g_{ij}(x, y)y^i y^j = y^i y_i$ for $y_i = g_{ij}y^j$.

If $\gamma^i_{jk}(x, y)$ are the "generalized" Christoffel symbols constructed using $g_{ij}(x, y)$, and $\gamma^i_{00}(x, y) = \gamma^i_{jk}(x, y)y^i y^j$, then the functions $N^i_j(x, y) = \dfrac{1}{2}\dot{\partial}_j(\gamma^i_{00})$, $\dot{\partial}_j := \dfrac{\partial}{\partial y^j}$ are the local coefficients of the nonlinear Cartan connection of F^n. For details see Ch. VIII in [7].

Using them, a new local basis $(\delta_i, \dot{\partial}_i)$, where $\delta_k = \partial_i - N_i^k \dot{\partial}_k$, on $\overset{\circ}{TM}$ is introduced. The dual of this basis is $(dx^i, \delta y^i = dy^i + N_k^i dx^k)$. If the quadratic form of matrix $(g_{ij}(x,y))$ is positive defined, then $G_S = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j$ is a Riemannian metric on the tangent manifold $\overset{\circ}{TM}$, called the Sasaki–Matsumoto lift of (g_{ij}) to $\overset{\circ}{TM}$. The linear operator F given in the local basis $(\delta_i, \dot{\partial}_i)$ as follows: $F(\delta_i) = -\dot{\partial}_i$, $F(\dot{\partial}_i) = \delta_i$, defines an almost complex structure on $\overset{\circ}{TM}$ and the pair (F, G_S) is an almost Kählerian structure on $\overset{\circ}{TM}$.

On $\overset{\circ}{TM}$ there exist two remarkable vector fields: $C = y^i \dot{\partial}_i$, called the Liouville vector field and $S = y^i \dot{\partial}_i$, which is the geodesic spray of F^n.

A framed $f-$structure is a natural generalization of an almost contact structure. It was introduced by S.I. Goldberg and K. Yano [3]. We recall its definition following [5, p.47].

Let N be a $(2n+s)-$dimensional manifold endowed with an endomorphism f of rank $2n$, of the tangent bundle, satisfying $f^3 + f = 0$. If there exist on N the vector fields (ξ_α) and the 1-forms y^α $(\alpha = 1, 2, ..., s)$ such that $\eta^\alpha(\xi_\beta) = \delta_\beta^\alpha$, $f(\xi_\alpha) = 0$, $\eta^\alpha \circ f = 0$, $f^2 = -I + \sum_\alpha \eta^\alpha \otimes \xi_\alpha$, where I is the identity automorphism of the tangent bundle, then N is said to be a framed $f-$manifold.

2 A framed $f-$structure on $\overset{\circ}{TM}$

Denote $\xi_1 = y^i \delta_i = S$ and $\xi_2 = y^i \dot{\partial}_j = C$. From the definition of F it follows

Lemma 2.1. $F(\xi_1) = -\xi_2$, $F(\xi_2) = \xi_1$.

We introduce the 1-forms $\eta^1 = \dfrac{y_i}{L^2} dx^i$ and $\eta^2 = \dfrac{y_i}{L^2} = \delta y^i$. These are globally defined on $\overset{\circ}{TM}$. By a direct calculation one gets

Lemma 2.2. $\eta^1 \circ F = \eta^2$, $\eta^2 \circ F = -\eta^1$.

Let be $G = \dfrac{1}{L^2} G_S$. One easily verifies

Lemma 2.3. *$\eta^1(X) = G(X, \xi_1)$, $\eta^2(X) = G(X, \xi_2)$, for every $X \in \mathcal{X}(\overset{\circ}{TM})$, the module of vector fields on $\overset{\circ}{TM}$.*

We define a tensor field of type $(1,1)$ on $\overset{\circ}{TM}$ by

$$f(X) = F(X) + \eta^1(X)\xi_2 - \eta^2(X)\xi_1, \ X \in \mathcal{X}(\overset{\circ}{TM}). \tag{2.1}$$

Theorem 2.1. *The ensemble $(f, (\xi_a), (\eta^b))$ $a, b, ... = 1, 2$ provides a framed $f-$structure on $\overset{\circ}{TM}$, that is the followings hold:*

(i) f *is of* rank $2n-2$ *and* $f^3 + f = 0$,

(ii) $\eta^a(\xi_b) = \delta^a_b,\ f(\xi_a) = 0,\ \eta^a \circ f = 0$,

(iii) $f^2(X) = -X + \eta^1(X)\xi_1 + \eta^2(X)\xi_2,\ X \in \mathcal{X}(\overset{\circ}{TM})$.

Proof. Using (2.1) and the Lemmas 2.1 and 2.2 one easily checks (ii) and (iii). Applying f on the equality (ii) one obtains the second part in (i). From the second equations in (ii), we see that $\operatorname{span}\{\xi_1, \xi_2\} \subseteq \operatorname{Ker} f$. If $X = X^k\delta_k + \dot{X}^k\dot{\partial}_k$ belongs to $\operatorname{Ker} f$ and it is not in $\operatorname{span}\{\xi_1, \xi_2\}$, we hve $X^i y_i = 0$ and $\dot{X}^i y_i = 0$ on $\overset{\circ}{TM}$, hence $X = 0$. Therefore, $\operatorname{Ker} f = \operatorname{span}\{\xi_1, \xi_2\}$ and $\operatorname{rank} f = 2n-2$, q.e.d.

Theorem 2.2. *The Riemannian metric* $G = \dfrac{1}{L^2}G_S$ *verifies*

$$G(fX, fY) = G(X,Y) - \eta^1(X)\eta^1(Y) - \eta^2(X)\eta^2(Y),\ X, Y \in \mathcal{X}(\overset{\circ}{TM}). \tag{2.2}$$

Proof. From (2.1) the following local expression of f is obtained

$$\begin{aligned} f(\delta_i) &= \left(-\delta^j_i + \frac{1}{L^2}y_i y^j\right)\dot{\partial}_j,\\ f(\dot{\partial}_i) &= \left(\delta^j_i - \frac{1}{L^2}y_i y^j\right)\delta_j. \end{aligned} \tag{2.3}$$

Using (2.3) one finds

$$\begin{aligned} &G(f(\delta_i), f(\delta_j)) = \frac{1}{L^2}\left(g_{ij} - \frac{1}{L^2}y_i y_j\right),\\ &G(f,\delta_i), f(\dot{\partial}_j) = 0,\\ &G(f(\dot{\partial}_i), f(\dot{\partial}_j)) = \frac{1}{L^2}\left(g_{ij} - \frac{1}{L^2}y_i y_j\right). \end{aligned}$$

From here easily follows (2.2). As (2.3) shows the operator f appears as a deformation of F similar with that studied in [1].

Remark 2.1. The metric G is homogeneous on the fibres of $\overset{\circ}{TM}$ while G_S is not. See [6].

Let us set $\phi(X,Y) = G(fX, Y)$ for $X, Y \in \mathcal{X}(\overset{\circ}{TM})$. Using Theorems 2.1, 2.2 one verifies

$$\phi(Y, X) = -\phi(X, Y). \tag{2.4}$$

Thus ϕ is a 2-form on $\overset{\circ}{TM}$.

Theorem 2.1 shows that the annihilator of ϕ is span$\{\xi_1, \xi_2\}$. A direct calculation gives $[\xi_2, \xi_1] = \xi_1$. Hence the distribution span$\{\xi_1, \xi_2\}$ is integrable even if ϕ is not closed. (The annihilator of a closed 2-form is always integrable.) A calculation in local coordinates leads to

$$\phi = d\eta^1 + \varphi, \text{ where } \varphi = \frac{1}{L^4} y_i y_j dx^i \wedge \delta y^j. \tag{2.5}$$

Thus ϕ is closed if and only if φ is closed and this happens under strong restrictions on the curvatures of the Cartan connection. Concluding, ϕ is in general an almost presymplectic structure on $\overset{\circ}{TM}$. Notice that $d\eta^1$ is a symplectic structure on $\overset{\circ}{TM}$. It appears as a deformation of the symplectic structure $\phi_S(X, Y) = G_S(FX, Y)$, $X, Y \in \mathcal{X}(\overset{\circ}{TM})$ since we have $d\eta^1 = \frac{1}{L^2}\phi_S + 2\varphi$.

3 An almost contact structure on the indicatrix bundle of F^n

The set $IM = \{(x, y) \in \overset{\circ}{TM} \mid L(x, y) = 1\}$ is called the indicatrix bundle of F^n. This set is a submanifold of dimension $2n - 1$ of $\overset{\circ}{TM}$. We show that the framed $f-$structure on $\overset{\circ}{TM}$ given by Theorem 2.2 induces an almost contact structure on $\overset{\circ}{TM}$. (This has to be compared with that from [4].)

It is well–known that $\xi_2 = y^i \dot{\partial}_i$ is normal to IM. We notice that it has the length 1 with respect to G. Thus the vector fields tangent to IM verify $G(X, \xi_2) = 0$. Let us restrict to IM the notions introduced above. Denote the restrictions putting a bar over that symbol. For $X, Y, ...$ vector fields which are tangent to IM we have:

- $\overline{\xi}_1 = \xi_1$ since ξ_1 is tangent to IM,
- $\eta^2 \equiv 0$ on IM since $\eta^2(X) = G(X, \xi_2)$,
- $\overline{G} = G_S$ because $L^2 = 1$ on IM,
- $\overline{f}(X) = F(X) + \overline{\eta}^2(X)\xi_2$ is an endomorphism of the tangent bundle of IM since $G(\overline{f}(X), \xi_2) = 0$.

We put $\overline{\xi} = \overline{\xi}_1$, $\overline{\eta} = \overline{\eta}^1$.

Theorem 3.1. *The ensemble $(\overline{f}, \overline{\xi}, \overline{\eta})$ provides an almost contact structure on IM, that is the followings hold:*

(i) $\overline{f}^3 + \overline{f} = 0$, $\operatorname{rank} \overline{f} = 2n - 2 = (2n - 1) - 1$

(ii) $\overline{\eta}(\overline{\xi}) = 1$, $\overline{f}(\overline{\xi}) = 0$, $\overline{\eta} \circ \overline{f} = 0$

(iii) $\overline{f}^2(X) = -X + \overline{\eta}(X)\overline{\xi}$, *for X a vector tangent to IM.*

Proof. All questions follows from those proved in Theorem 2.2 by virtue of the above considerations on the restrictions to IM of the ensemble $(f, (\xi_a), (\eta^a))$, $a = 1, 2$.

From Theorem 2.2 it follows

Theorem 3.2. *The Riemannian metric G_S verifies*

$$G_S(\overline{f}X, \overline{f}Y) = G_S(X, Y) - \overline{\eta}(X)\overline{\eta}(Y), \tag{3.1}$$

for X, Y vectors tangent to IM.

One checks that $(\delta_i, \overline{f}\delta_j)$, $j = 1, ..., n-1$, is a local frame on a neighborhood with $y^n \neq 0$ on IM. As the points $(x, 0)$ are outside of IM one always may consider such a local frame.

Let $\Omega(X, Y) = G_S(\overline{f}X, Y)$ be the 2-form usually associated to an almost contact structure.

By a direct calculation one gets

$$\begin{aligned} d\overline{\eta}(\delta_i, \delta_j) &= 0 = \Omega(\delta_i, \delta_j) \\ d\overline{\eta}(\delta_i, \overline{f}\delta_j) &= g_{ij} - y_i y_j = \Omega(\delta_i, f\delta_j) \\ d\overline{\eta}(\overline{f}\delta_i, \overline{f}\delta_j) &= 0 = \Omega(\overline{f}\delta_i, \overline{f}\delta_j), \end{aligned} \tag{3.2}$$

in other words $\Omega = d\overline{\eta}$.

In all these calculation we have used the Cartan connection of the Finsler space F^n.

Thus we have

Theorem 3.3. *Let F^n be endowed with the Cartan connection. Then the structure $(\overline{f}, \overline{\xi}, \overline{\eta}, G_S)$ is a contact Riemannian structure on IM.*

The structure $(\overline{f}, \overline{\xi}, \overline{\eta})$ is called *normal* if $N = N_{\overline{f}} + d\overline{\eta} \otimes \overline{\xi} = 0$, where $N_{\overline{f}}$ is the Nijenhuis tensor field of $\overline{f}$. And it is said to be Sasakian if it is normal and $\Omega = d\overline{\eta}$. Again by a direct calculation one find $N(\delta_i, \delta_j) = (y_i\delta_j^h - y_j\delta_i^h - R^h{}_{ij})\dot{\partial}_h$, and the vanishing of this term implies the vanishing of $N(\overline{f}\delta_i, \delta_j)$ and $N(\overline{f}\delta_i, \overline{f}\delta_j)$. But $N(\delta_i, \delta_j) = 0$ is equivalent with

$$R^h{}_{ij} = y_i\delta_j^h - y_j\delta_i^h, \tag{3.3}$$

where $R^h{}_{ij} = R_k{}^h{}_{ij}y^k$ and $R_k{}^h{}_{ij}$ is the $hh-$curvature of the Cartan connection.

The equality (3.3) takes also the form

$$R_{ihk} = g_{ik}y_h - g_{ih}y_k, \tag{3.4}$$

which says that F^n is of constant curvature 1.

Thus we have

Theorem 3.4. *Let F^n endowed with Cartan connection. Then the structure $(\overline{f}, \overline{\xi}, \overline{\eta}, G_S)$ on IM is a Sasakian structure if and only if F^n is of constant curvature* 1.

For Finsler spaces of constant curvature we refer to [2]. It seems that the almost contact structure given in Theorem 3.1 is very close with that obtained in [4]. They have the same properties (Theorems 3.3 and 3.4).

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Faculty of Mathematics
"Al. I. Cuza" University of Iasi

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SOME RIEMANNIAN ALMOST PRODUCT STRUCTURES ON TANGENT MANIFOLD

BY

M. ANASTASIEI[3]

Abstract

The tangent manifold TM of a smooth i.e. C^∞, paracompact manifold M, fibered over M by the natural projection τ, carries an integrable distribution $\operatorname{Ker}\tau_*$, called *vertical distribution*. If one takes a supplementary distribution of it, called *horizontal*, an almost product structure P on TM appears. One endows the vertical distribution with a Riemannian metric g. Then g can be prolonged to a Riemannian metric G on TM in such a way that the pair (P, G) becomes a Riemannian almost product structure. In this paper we propose a deformation of P suggested the almost complex case, [1]. This produces six new Riemannian almost product structures. Some properties of these structures are pointed out. The particular case when g is the vertical lift of a Riemannian metric on M is considered.

MSC2000 : 53 C 15

1 A standard Riemannian almost product structure on TM

Let M be a smooth i.e. C^∞ paracompact manifold of dimension n with local coordinates (x^i), $i, j, k... = 1, ..., n$. Denote by TM its tangent manifold with local coordinates (x^i, y^i) and projection $\tau : TM \to M$. It is known that TM is also paracompact. Let $V_uTM = \operatorname{Ker}\tau_{*,u}$ for $u \in TM$. Then $u \to V_uTM$ is an integrable distribution on TM, called *vertical distribution* and $VTM = \bigcup\limits_u V_uTM$ is the vertical bundle over TM.

Let HTM be a vector bundle over TM which is supplementary to VTM. Such a vector bundle, called *horizontal*, always exists since TM is paracompact. It is said also that it defines a nonlinear connection on TM. Thus we have the decomposition

$$T_uTM = H_uTM \oplus V_uTM. \tag{1.1}$$

[3]Partially supported by CNCSIS Bucureşti, Romania

The projectors h and v produced by the direct sum (1.1) provide an almost product structure P (AP–structure for brevity), given by $P = h - v$. Thus we notice

$$P^2 = I,\ h = \frac{1}{2}(I + P),\ v = \frac{1}{2}(I - P). \tag{1.2}$$

The horizontal and vertical subspaces in $T_u TM$ are eigenspaces of P corresponding to the eigenvalues $+1$ and -1, respectively.

The vertical distribution is locally spanned by $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}\cdot$ Looking for a basis (δ_i) in $H_u TM$ in such a way that $\tau_*(\delta_i) = \partial_i := \dfrac{\partial}{\partial x^i}$, one finds that

$$\delta_i = \partial_i - N_i{}^j(x, y)\dot{\partial}_j$$

(the sign "$-$" is for convenience), where the functions $(N_i{}^j(x, y))$ transform by a change of coordinates $(x^i, y^i) \to (\widetilde{x}^i, \widetilde{y}^i)$ as follows:

$$\widetilde{N}_i{}^k \partial_j \widetilde{x}^i = \partial_h \widetilde{x}^k \cdot N_j{}^h - \partial_j \partial_h \widetilde{x}^k . y^h. \tag{1.3}$$

In terminology from [5], $(N_i{}^j)$ define a nonlinear connection.

The basis $(\delta_i, \dot{\partial}_i)$ is adapted to the decomposition (1.1). Its dual is $(dx^i, \delta y^i)$ for $\delta y^i = dy^i + N_k{}^i(x, y)dx^k$. In the adapted basis P takes the form

$$P(\delta_i) = \delta_i,\ P(\dot{\partial}_i) = -\dot{\partial}_i, \tag{1.4}$$

i.e. it has the matrix $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$.

In general, the horizontal distribution is not integrable. The nonintegrability is measured by the functions $R^k{}_{ij}(x, y)$ from

$$[\delta_i, \delta_j] = R^k{}_{ij}(x, y)\dot{\partial}_k. \tag{1.5}$$

The functions $(R^k{}_{ij})$ behave like the components of a tensor of M i.e. they define a d–tensor field. These functions are regarded as the curvature of the nonlinear connection $(N_j{}^i)$. We notice for the later use

$$[\delta_i, \dot{\partial}_j] = \dot{\partial}_j(N_i{}^k)\dot{\partial}_k. \tag{1.6}$$

One says that P is integrable if the horizontal and vertical distributions are integrable. Thus we have

Theorem 1.1. *The AP-structure P is integrable if and only if $R^k{}_{ij}(x, y) = 0$,*
equivalently the nonlinear connection $(N_j{}^i)$ is without curvature.

Now let us endow the vertical bundle over TM with a Riemannian metric g. We may do this since M is paracompact. The local components of

g, given by $g_{ij}(x,y) = g(u)(\dot{\partial}_i, \dot{\partial}_j)$, $u \in TM$ define a d–tensor field of type $(0,2)$, symmetric and positive defined. In fact g is nothing but a generalized Lagrange metric introduced by R. Miron and studied by him and his co-workers, see [, 5, Ch. X–XII]. The Riemannian metric g may be extended to a Riemannian metric G on TM given in the form

$$G(u) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^i, \ u = (x,y) \in TM. \tag{1.7}$$

It is clear that H_uTM and V_uTM are orthogonal with respect to G. One easily see that $G(PX, PY) = G(X,Y)$ for any vector fields X, Y on TM. Thus we have

Theorem 1.2. *The pair (P, G) is a Riemannian AP–structure on TM.*

Let D be a linear connection on TM with the torsion T.

If P is parallel with respect to D i.e. $D_X P = 0$, then the Nijenhuis tensor field associated to P takes the form

$$N_P(X,Y) = T(X,Y) + PT(X, PY(-PT(PX,Y) - T(PX,PY)$$

for X, Y vector fields on TM. This form proves

Theorem 1.3. *If the Levi–Civita connection of G makes P parallel, then the AP–structure P is integrable.*

2 Deformations of the Riemannian AP–structure P

We set $y_i = g_{ij}(x,y)y^j$ and consider the following deformations of P

$$\begin{aligned} P_d(\delta_i) &= (\alpha\delta_i^k + \beta y_i y^k)\delta_k, \\ P_d(\dot{\partial}_i) &= (\gamma\delta_i^k + \delta y_i y^k)\dot{\partial}_k, \end{aligned} \tag{2.1}$$

for $\alpha, \beta, \gamma, \delta$ functions on TM, to be determined in such a way that $P_d^2 = I$ and $G(P_d\cdot, P_d\cdot) = G(\cdot,\cdot)$. This deformation is suggested by the almost complex case, see [1]. The condition $P_d^2 = I$ shows that $\alpha, \beta, \gamma, \delta$ have to be solutions of the following system of equations

$$\begin{aligned} &\alpha^2 = 1, \quad \beta(2\alpha + \beta F^2) = 0, \\ &\gamma^2 = 1, \quad \delta(2\gamma + \delta F^2) = 0, \end{aligned} \tag{2.2}$$

for $F^2 = g_{ij}(x,y)y^i y^j$.

This system of equations has sixteen solutions. Inserting them in (2.1) one finds, leaving aside the trivial AP–structures $\pm I$, fourteen AP–structures from which seven are essential, the other seven differing by a sign from the previous ones. If we put $A = (A_i^j) = \left(\delta_i^j - \dfrac{2}{F^2}y_i y^j\right)$, these AP–structures are given in matrix form as follows

$$
P_0 \equiv P = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},\ P_1 = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix},\ P_2 = \begin{pmatrix} I_n & 0 \\ 0 & -A \end{pmatrix},
$$

$$
P_3 = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix},\ P_4 = \begin{pmatrix} A & 0 \\ 0 & -I_n \end{pmatrix}, \tag{2.3}
$$

$$
P_6 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},\ P_6 = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}.
$$

The condition $G(P_d\cdot, P_d\cdot) = G(\cdot,\cdot)$ does not impose any restriction on $\alpha, \beta, \gamma, \delta$ previously determined. Thus we have

Theorem 2.1. *The pairs (P_α, G), $\alpha = 0, ..., 6$ are seven Riemannian AP–structures on TM.*

Remark 2.1. The set $\{I, P_0, P_1, ..., P_6\}$ has a group structure given by the table

	I	P_0	P_1	P_2	P_3	P_4	P_5	P_6
I	I	P_0	P_1	P_2	P_3	P_4	P_5	P_6
P_0	P_0	P_1	P_2	P_1	P_4	P_3	P_6	P_5
P_1	P_1	P_2	I	P_0	P_5	P_6	P_3	P_4
P_2	P_2	P_1	P_0	I	P_6	P_5	P_4	P_3
P_3	P_3	P_4	P_5	P_6	I	P_0	P_1	P_2
P_4	P_4	P_3	P_6	P_5	P_0	I	P_2	P_1
P_5	P_5	P_6	P_3	P_4	P_1	P_2	I	P_0
P_6	P_6	P_5	P_4	P_3	P_2	P_1	P_0	I

This group is commutative. Its proper subgroups are $\{I, P_\alpha\}$ for $\alpha = 0, 1, ..., 6$, and $\{I, P_0, P_1, P_2\}$, $\{I, P_0, P_3, P_4\}$, $\{I, P_1, P_4, P_6\}$, $\{I, P_0, P_5, P_6\}$, $\{I,\ P_1,\ P_3,\ P_5\}$, $\{I, P_2, P_3, P_6\}$, $\{I, P_2, P_4, P_5\}$. The last seven are isomorphic with the Klein group. The group can be also seen as a Burnside group $B(2,3)$ generated by $\{P_0, P_3, P_5\}$

Let $h_\alpha = \dfrac{1}{2}(I + P_\alpha)$ and $V_\alpha = \dfrac{1}{2}(I - P_\alpha)$ be the projectors defined by P_α and let us set $H_\alpha = \operatorname{Ker} v_\alpha$, $v_\alpha = \operatorname{Ker} h_\alpha$, for $\alpha = 0, 1, ..., 6$ with $h_0 = h$, $v_0 = v$, $H_0 = H$, $V_0 = V$.

For identifying the distributions H_α and V_α, $\alpha = 1, ..., 6$, we consider the vector fields $C = y^i \dot{\partial}_i$ and $S = y^i \delta_i$ and denote by the same letters the 1-dimensional distributions defined by them.

Furthermore, we denote by $C^\perp$ the orthocomplement of C in V, that is, $C^\perp = \{A^i \dot{\partial}_i \mid g_{ij} y^i A^j = A^j y_j = 0\}$ and by $S^\perp$ the orthocomplement of S in H, that is, $S^\perp = \{X^i \delta_i \mid g_{ij} y^i X^j = X^j y_i = 0\}$. With this notations the following result holds.

Theorem 2.2. *The distributions defining P_α are as follows:*

$$
\begin{cases} H_0 = H \\ V_0 = V \end{cases}
\begin{cases} H_1 = H \oplus C^\perp \\ V_1 = C \end{cases}
\begin{cases} H_2 = H \oplus C \\ V_2 = C^\perp \end{cases}
\begin{cases} H_3 = V \oplus S^\perp \\ V_3 = S \end{cases}
$$

$$\begin{cases} H_4 = S^\perp \\ V_4 = V \oplus S \end{cases} \quad \begin{cases} H_5 = [S]^\perp \oplus [C]^\perp \\ V_5 = [S] \oplus [C] \end{cases} \quad \begin{cases} H_6 = S^\perp \oplus C^\perp \\ V_6 = S \oplus C \end{cases}$$

Proof. For $\alpha = 1$, we have $h_1(\delta_i) = \delta_i$, $h_1(\dot{\partial}_i) = \left(\delta_i^k - \frac{1}{F^2} y_i y^k\right) \dot{\partial}_k$, $v_i(\delta_i) = 0$, $v(\dot{\partial}) = \frac{1}{F^2} y_i y^k \dot{\partial}_k$. From these equations one gets

$$\begin{aligned} V_1 &= \operatorname{Ker} h_1 = \left\{ X^i \delta_i + A^i \dot{\partial}_i \mid X^i \delta_i + \left(\delta_i^k - \frac{1}{F^2} y_i y^k\right) A^i \dot{\partial}_k = 0 \right\} = \\ &= \left\{ X^i \delta_i + A^i \dot{\partial}_i \mid X^i = 0,\ A^k = \frac{1}{F^2}(A^i y_i) y^k \right\} = C \text{ and} \\ H_1 &= \operatorname{Ker} v_1 = \left\{ X^i \delta_i + A^i \dot{\partial}_i \mid A^i y_i = 0 \right\} = H \oplus C^\perp. \end{aligned}$$

Similarly, one finds the other distributions.

A study of the integrability of the distributions V_α, $\alpha = 0, 1, ..., 6$ gives

Theorem 2.3.

1) *The distributions $V_0 = V, V_1, V_3$ are always integrable.*
2) *The distribution V_2 is integrable if there exists a real function L on TM such that $g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L$.*
3) *The distribution V_5 is integrable if the nonlinear connection (N_j^i) is positively homogeneous of degree 1.*
4) *The distributions V_4 and V_6 are never integrable.*

Proof. 1) We have noticed that V is integrable. The distributions V_1 and V_3 are 1-dimensional, hence integrable.

2) Let $A = A^i \dot{\partial}_i$ and $B = B^j \dot{\partial}_j$ in $C^\perp$ i.e. $A^i y_i = 0$, $B^i y_j = 0$. Then $[A, B] \in C^\perp$ if and only if $A^i \dot{\partial}_i (B^j) y_j - B^i \dot{\partial}_i (A^j) y_j = 0$, equivalently $A^j B^i \dot{\partial}_i (y_j) - A^i B^j \dot{\partial}_i (y_j) = 0$. This condition identically holds if $\dot{\partial}_i g_{jk} = \dot{\partial}_j g_{ik}$.

3) It is clear that the distribution $S \oplus C$ is integrable if $[C, S]$ belongs to it. We hve $[C, S] = S + y^j (N_j^k - y^i \dot{\partial}_i (N_j^k)) \dot{\partial}_k = S$, if the functions $(N_j^k(x, y))$ are positively homogeneous of degree 1 with respect to (y^i).

4) A direct calculation.

Remark 2.2. If $g_{ij}(x, y)$ is the metric tensor of a Finsler space and $(N_j^i(x, y))$ is the Cartan nonlinear connection, the hypothesis in 2) and 3) of Theorem 2.3 are satisfied and so in this case the distributions V_0, V_1, V_3, V_5 are integrable.

The distributions H_α, $\alpha = 0, 1, ..., 6$ are not integrable or they are so in very strong conditions. We renounce to write down such conditions. A

Riemannian AP–structure is integrable if the both distributions defining it are integrable. From the above it follows the Riemannian AP–structures P_4 and P_6 are never integrable. The others are integrable only under some strong conditions on (g_{ij}) and (N^i_j).

The Riemannian AP–structures were classified by A.M. Naveira [6]. Modulo a duality there exists thirty–six different classes described by conditions on ∇h_α, where ∇ denotes the Levi–Civita connection of G. See also [4].

From [2] it follows that the Levi–Civita connection ∇ can be taken in the form

$$\begin{aligned} &\nabla_{\delta_k}\delta_j = F^i_{jk}\delta_i + A^a_{jk}\dot{\partial}_a, \quad \nabla_{\dot{\partial}_b}\delta_j = \widetilde{C}_j{}^i{}_b\delta_i + E_j{}^a{}_b\dot{\partial}_a, \\ &\nabla_{\delta_k}\dot{\partial}_b = L^a_{bk}\dot{\partial}_a + D^i_{bk}\delta_i, \quad \nabla_{\dot{\partial}_b}\dot{\partial}_c = C^a_{cb}\dot{\partial}_a + B^i_{cb}\delta_i, \end{aligned} \tag{2.1}$$

with the connection coefficients given by

$$\begin{aligned} &F^i_{jk} = \frac{1}{2}g^{ih}(\delta_j g_{hk} - \delta_k g_{jh} - \delta_h g_{jk}), \\ &A^a_{jk} = \frac{1}{2}(-R^a_{jk} - g^{ab}\dot{\partial}_b g_{jk}), \\ &\widetilde{C}_j{}^i{}_b = D^i_{bj} = \frac{1}{2}g^{ih}(\dot{\partial}_b g_{jh} + g_{bc}R^c{}_{hj}), \\ &E^a_{ib} = \frac{1}{2}g^{ac}g_{bc\|i}, \ L^a_{bi} = \dot{\partial}_b N^a_i + \frac{1}{2}g^{ac}g_{bc\|i}, \\ &B^k_{ab} = -\frac{1}{2}g^{kj}g_{ab\|j}, \ C^a_{bc} = \frac{1}{2}g^{ad}(\dot{\partial}_b g_{dc} + \dot{\partial}_c g_{bd} - \dot{\partial}_d g_{bc}). \end{aligned} \tag{2.2}$$

Here $g_{bc\|i}$ denotes the h–covariant derivative with respect to the Berwald connection i.e.

$$g_{bc\|i} = \delta_i g_{bc} - (\dot{\partial}_b N^d_i)g_{dc} - (\dot{\partial}_c N^d_i)g_{bd}.$$

We do not classify P_α here but we remark that P_4 and P_6 cannot be in Naveira's class $\mathcal{P}$. Indeed, the conditions $\nabla P_\alpha = 0$ characterizing $\mathcal{P}$ implies in virtue of Theorem 1.3 that P_α should be integrable. But P_4 and P_6 are never integrable.

A distribution $\mathcal{D}$ is geodesically invariant if all geodesics with initial vector in $\mathcal{D}$ remain tangent to D for all time. As it was proved in [4], a distribution $\mathcal{D}$ is geodesically invariant if and only if for any sections X, Y of $\mathcal{D}$, the symmetric product $X : Y = \nabla_X Y + \nabla_Y X$ is again a section of D. See also [1]. Using (2.1) and (2.2) one gets

Theorem 2.4.

1) *The distribution H is geodesically invariant if and only if $g_{ij\|k} = 0$.*

2) *The distribution V is geodesically invariant if and only if $\dot{\partial}_k g_{ij} = 0$.*

The first condition in Theorem 2.4 tells us that we have to assume no dependence on $y = (y^i)$ in (g_{ij}). In this case $g(g_{ij})$ reduces to a Riemannian metric on M. If we continue to work with an arbitrary nonlinear connection (N^i_j), the second condition in Theorem 2.4 is not verified. But if we

take $N_j^i(x,y) = \gamma_{jk}^i(x)y^k$, where (γ_{jk}^i) are Christoffel symbols derived from $(g_{ij}(x))$, then the condition $g_{jk\|h} = 0$ reduces to $\partial_h g_{jk} - \gamma_{hj}^i g_{ik} - \gamma_{hk}^i g_{ji} = 0$ which obviously holds. We consider this case in what follows. Thus we may state

Corollary 2.1. *Let (M,g) be a Riemannian manifold. One considers TM endowed with the nonlinear connection $N_j^i(x,y) = \gamma_{jk}^i(x)y^k$ and one defines the Riemannian metric G with this nonlinear connection (G becomes the Sasaki metric of g). Then the distributions V and H are geodesically invariant.*

We associate to every P_α, $\alpha = 0,1,...,6$, the symmetric tensor field ϕ_α defined by

$$\phi_\alpha(X,Y) = G(P_\alpha X, Y),\ X,Y \in \mathcal{X}(TM). \tag{2.6}$$

Using the matrix form of P_α one obtains

$$\begin{aligned}
\phi_1(u) &= g_{ij}dx^i \otimes dx^j - g_{ij}\delta y^i \otimes \delta y^j,\\
\phi_1(u) &= g_{ij}dx^i \otimes dx^j + \left(g_{ij} - \frac{2}{F^2}y_iy_j\right)\delta y^i \otimes \delta y^j,\\
\phi_2(u) &= g_{ij}dx^i \otimes dx^j - \left(g_{ij} - \frac{2}{F^2}y_iy_j\right)\delta y^i \otimes \delta y^j,
\end{aligned}$$

and so on.

As $\left(g_{ij} - \frac{2}{F^2}y_iy_j\right)$ is an invertible matrix, with the inverse $\left(g^{jk} - \frac{2}{F^2}y^iy^j\right)$, it follows that ϕ_α, $\alpha = 0,1,...,6$ are pseudo–Riemannian metrics on TM.

If we look at the first terms in ϕ_0, ϕ_1, ϕ_2, it appears as obvious

Theorem 2.5. *The maps $\tau : (TM, \phi_\alpha) \to (M,g)$, $\alpha = 0,1,2$ are Riemannian submersions.*

A fundamental tensor field of the submersions τ_α is

$$S_\alpha(X,Y) = h_\alpha \nabla_{v_\alpha X} v_\alpha Y + v_\alpha \nabla_{v_\alpha X} h_\alpha Y,\ X,Y \in \mathcal{X}(TM)$$

and the submersion τ_α is called *totally geodesic* if S_α identically vanishes. Here $\alpha = 0,1,2$.

A tedious calculation in which the identity $y_iy^k\dot{\partial}\left(\frac{1}{F^2}y_iy^h\right) = 0$ is used proves

Theorem 2.6. *The Riemannian submersions $\tau_\alpha : (TM, \phi_\alpha) \to (M,g)$ are totally geodesic, $\alpha = 0,1,2$.*

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SYMPLECTIC CONNECTIONS IN LAGRANGE GEOMETRY

BY

M. ANASTASIEI

Abstract

A Lagrangian structure and, in particular, a Finslerian or a Riemannian structure on a manifold M induces a symplectic structure on TM. We investigate the linear connections on TM depending on the Lagrangian structure only, which are compatible with this symplectic structure and have no torsion.
MSC 2000: 53C60,53D99

Introduction

On TM we have the vertical bundle as the kernel of the differential of the projection $TM \to M$. We take a supplement of it, that is, a horizontal bundle or a nonlinear connection and we consider the natural almost complex product F on TM associated to these bundles. We show in the first section of this paper that the symplectic structures on TM having the horizontal and vertical bundles as Lagrangian subbundles and being compatible with F are essentially induced by a Lagrangian structure on M. In the second section we state precisely the symplectic structure Ω_L induced by a Lagrangian structure L. In the third section we are interested in linear connections on TM which are compatible with Ω_L and have no torsion, called symplectic connections. First, we notice that the liner Cartan connection of L is compatible with Ω_L but has torsion. By a deformation of it we find a set of symplectic connections from which set one depending on L only is single out. In the case that the horizontal distribution is integrable, a set of symplectic connections preserving by parallelism the vertical and horizontal bundles is found. I. Vaisman discovered [3] three classes of symplectic connections: flat, Ricci flat and with reducible curvature. Among the symplectic connections that we have found, two flat symplectic connections and a Ricci flat one are pointed out in section 4. We refer to the monograph [2] for notations and terminology.

1 Symplectic structures on TM

We shall work in the category of real, smooth, i.e. C^∞ and finite dimensional manifolds. Let M be a manifold of dimension n and $\tau : TM \to M$ its tangent bundle. Let $(U, (x^i))$, $i = 1, 2, ..., n$ be a coordinate chart on M. Then $(\tau^{-1}(U), (x^i \circ \tau, y^i))$, where (y^i) are the components of a tangent vector v_x, $x \in U$, in the natural basis $\partial_i := \dfrac{\partial}{\partial x^i}$ of T_xM, is a coordinate chart on TM. The indices $i, j, k...$ will range from 1 to n and the Einstein convention on summation will be used.

Let $\tau_* : TTM \to TM$ be the differential of τ. The union of $V_uTM := \ker \tau_{*,n}$ for $u \in TM$ defines the vertical bundle over TM. We may thought it as a distribution on TM called vertical distribution. This is locally spanned by $\dot{\partial}_i := \dfrac{\partial}{\partial y^i}$, hence it is integrable. Thus we may speak about vertical foliation whose leaves are T_xM, $x \in M$. A *non–linear connection* N is a subbundle HTM of TTM, called horizontal, that is supplementary to the vertical bundle, i.e. the following decomposition holds

$$TTM = VTN \oplus HTM \text{ (Whitney's sum)} \tag{1.1}$$

We also view the horizontal subbundle as a distribution $u \to H_uTM$ called the *horizontal distribution* on TM.

Locally, we shall use the *adapted bases* $(\delta_i, \dot{\partial}_i)$, where

$$\delta_i = \partial_i - N_i^k(x, y)\dot{\partial}_k m \tag{1.2}$$

span the horizontal distribution, and their dual cobases $(dx^i, \delta y^i)$, where

$$\delta y^i = dy^i + N_k^i(x, y)dx^k. \tag{1.3}$$

The functions (N_k^i) are called the local coefficients of the non–linear connection N. If these functions are linear with respect to (y^i), that is, $N_k^i(x, y) = \Gamma_{kj}^i(x)y^j$, it comes out that $(\Gamma_{kj}^i(x))$ are the local coefficients of a linear connection on M.

The tensor fields on TM get a natural multiple grading induced by (1.1). When this is made explicit by the use of the adapted bases and their dual cobases, the coefficients of the components are functions depending on (x, y) but transform under a change of coordinates on TM as tensors on M. it is said in [2] that these components or their coefficientsare d-tensor fields on TM. here d is for "distinguished". In particular, for the spaces of differential forms we have

$$\wedge^k(TM) = \oplus_{p+q=k} \wedge^{pq}(TM), \tag{1.4}$$

where p is the V-degree and q is the H-degree. Thus any 2-form Ω on TM can be written as

$$\Omega = \frac{1}{2}b_{ij}(x, y)dx^i \wedge dx^j + a_{ij}(x, y)dx^i \wedge \delta y^i + \frac{1}{2}c_{ij}(x, y)\delta y^i \wedge \delta y^j, \tag{1.5}$$

with $b_{ij} = -b_{ji},\ c_{ij} = -c_{ji}$. Each term in (1.5) is a distinguished 2-form on TM. The coefficients $a_{ij}(x, y)$, $b_{ij}(x, y)$, $c_{ij}(x, y)$ transform under a change

of coordinates on TM as the components of tensors on M, the last two being skew symmetric.

Let us suppose that Ω given by (1.5) defines a symplectic structure o TM. From

$$\begin{aligned} \Omega(\delta_i, \delta_j) &= b_{ij}, & \Omega(\delta_i, \dot{\partial}_j) &= a_{ij}, \\ \Omega(\dot{\partial}_i, \delta_j) &= -a_{ji}, & \Omega(\dot{\partial}_i, \dot{\partial}_j) &= c_{ij}, \end{aligned} \tag{1.6}$$

it comes out that the vertical (horizontal) bundle is a Lagrangian subbundle with respect to Ω if and only if $c_{ij} = 0$ ($b_{ij} = 0$). In the sequel we shall be interested only in symplectic structures on TM that make the vertical and horizontal bundles the Lagrangian subbundles of TTM. Thus we consider only the symplectic structures on TM given by the 2-forms

$$\Omega = a_{ij}(x, y) dx^i \wedge \delta y^i, \tag{1.7}$$

satisfying the conditions

$$\det(a_{ij}(x, y)) \neq 0 \Longleftrightarrow \Omega \text{ is nondegenerate}, \tag{1.8}$$

$$\begin{aligned} &\textstyle\sum_{(ijk)} a_{ih} R^h{}_{jk} = 0,\ \delta_i a_{jk} + a_{ih} \dot{\partial}_k N^h_j = \delta_j a_{ik} + a_{jk} \dot{\partial}_k N^h_i, \\ &\dot{\partial}_k a_{ij} = \dot{\partial}_j a_{ik}, \end{aligned} \tag{1.9}$$

where

$$[\delta_j, \delta_k] = R^h{}_{jk} \dot{\partial}_h,\ R^h{}_{jk} = \delta_k N^h_j - \delta_j N^h_k. \tag{1.10}$$

The eqs. (1.9) are equivalent with $d\Omega = 0$. The functions $(R^h{}_{jk}(x, y))$ define a d-tensor of type $(1, 2)$. It vanishes if and only if the horizontal distribution is integrable.

Now we consider the almost complex structure F on TM defined by

$$F(\delta_i) = -\dot{\partial}_i,\ F(\dot{\partial}_i) = \delta_i. \tag{1.11}$$

Let $\chi(TM)$ the set of vector fields on TM. It is easy to check

Proposition 1.1. *For $X, Y \in \chi(TM)$ we have*

$$\Omega(FX, FY) = \Omega(X, Y), \tag{1.12}$$

if and only if $a_{ij} = a_{ji}$.

We confine ourselves to the case when Ω from (1.7) satisfies (1.12). We put $a_{ij} = -g_{ij}$ with $g_{ij} = g_{ji}$ and we write Ω in the form

$$\Omega = g_{ij}(x, y) \delta y^i \wedge dx^j. \tag{1.13}$$

The d-tensor field $g = g_{ij}(x, y) \delta y^i \otimes \delta y^j$ with $\det(g_{ij}) \neq 0$ and such that the quadratic form $g_{ij} \xi^i \xi^j$, $(\xi^i) \in \mathbb{R}^n$, has constant signature, is called a

generalized Lagrange metric, shortly a GL-metric, [2]. One may consider also the d-tensor field $g_{ij}9x,y)dx^i \otimes dx^j$ which summed with g gives a metrical structure on TM:

$$G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{1.14}$$

One easily verifies

Proposition 1.2 *For every $X, Y \in \chi(TM)$ one has*

$$G(X,Y) = \Omega(X, FY), \tag{1.15}$$

$$G(FX, FY) = G(X,Y). \tag{1.16}$$

Thus the pair (F, G) is an almost Hermitian structure o TM and Ω appears as its fundamental 2-form. As $d\Omega = 0$, we have that (F, G) is an almost Kähler structure. It reduces to a Kähler structure if and only if ${R^h}_{jk} = 0$ and $\dot\partial_k n^i_h = \dot\partial_h N^i_k$, cf. [2], Ch.7.

The functions $(g_{ij}(x,y))$ have to satisfy the conditions

$$\begin{array}{l} \sum_{(ijk)} R_{ijk} = 0,\ \delta_i g_{jk} + g_{ih}\dot\partial_k N^h_j = \delta_j g_{ik} + g_{jh}\dot\partial_k N^h_i, \\ dot\partial_k g_{ij} = \dot\partial_j g_{ik},\ \text{for } R_{ijk} := g_{ih}{R^h}_{jk}, \end{array} \tag{1.9$'$}$$

in order that $d\Omega = 0$ for Ω given by (1.13).

The third equality in (1.9)$'$ holds if and only if

$$g_{ij}(x,y) = \frac{1}{2}\dot\partial_i\dot\partial_j L(x,y),\ \text{for some function } L \text{ on } TM. \tag{1.17}$$

We shall take the assumption (1.17) for the rest of this paper.

2 Lagrangian symplectic structures on TM

We call a Lagrangian structure on M a regular Lagrangian on TM, that is a function $L : TM \to R$ such that the matrix $(g_{ij}(x,y))$ given by (1.17) has $\det(g_{ij}) \neq 0$ and the quadratic form $g_{ij}(x,y)\xi^i\xi^j$, $\xi \in \mathbb{R}^n$, is of constant signature on TM. The pair (M, L) is called a Lagrange manifold. We send to the monograph [2] for the geometry of these manifolds. It is known that a Lagrangian structure determines a non–linear connection. This can be constructed as follows, [2, Ch.IX]. The Euler–Lagrange equations for L take the form

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0, \tag{2.1}$$

The functions $(-2G^i 9x,y))$ are the coefficients of a semispray (second order differential equation) on M and one proves that

$$N^i_j(x,y) = \dot\partial_j G^i(x,y), \tag{2.3}$$

are the local coefficients of a non–linear connection $N_L \circ TM$.

Now we may consider the adapted bases and their dual cobases with respect to N_L. We keep the notations from the first section but we refer now to N_L only.

Thus for the Lagrange manifold (M, L) we have $g_{ij}(x, y)$ given by (1.17) and $(N^i_j(x, y))$ given by (2.3). The symplectic structure

$$\Omega_L = g_{ij}(x, y)\delta y^i \wedge dx^j, \tag{2.4}$$

will be called a Lagrangian symplectic structure.

That Ω_L is indeed a symplectic structure and not only an almost symplectic one it follows from

Proposition 2.1. $\Omega_L = d\omega_L$ *for* $\omega_L = \dfrac{1}{2}(\dot{\partial}_j L)dx^j$.

Proof. We have $d\omega_L = \dfrac{1}{2}(\partial_i \dot{\partial}_j L)dx^i \wedge dx^j + g_{ij}dy^i \wedge dx^j$. Inserting here $dy^i = \delta y^i - N^i_k dx^k$, one gets

$$d\omega_L = \left(\frac{1}{2}\partial_i \dot{\partial}_j L - g_{kj}N^k_i\right) dx^i \wedge dx^j + g_{ij}\delta y^i \wedge dx^j = \Omega_L$$

because of the symmetry in the indices i, j of $A_{ij} = \dfrac{1}{2}\partial_i \dot{\partial}_j L - g_{kj}N^k_i$. Indeed, a direct calculation gives

$$4A_{ij} = (\partial_i \dot{\partial}_j L + \dot{\partial}_i \partial_j L) - 2y^s \partial_s g_{ij} + 4G^k \dot{\partial}_k g_{ij}, \text{ q.e.d.}$$

On the other hand the condition $d\Omega_L = 0$ is equivalent with $(1.9)'$ written for (g_{ij}) given by (1.17) and (N^j_i) given by (2.3). By Proposition 2.1 the conditions $(1.9)'$ become identities.

3 Symplectic connections for (TM, Ω_L)

A linear connection ∇ on TM endowed with Ω_L will be called almost symplectic if $\nabla\Omega_L = 0$. The term of symplectic connection is reserved for those almost symplectic connections which have no torsion. It is known that any symplectic manifold admits infinitely many almost symplectic connections and infinitely many symplectic connections. See [3] for a clear review of this matter.

For a Lagrange manifold (M, L), besides the non–linear connection N_L we have a canonical linear connection determined by L only, called the linear Cartan connection. We recall it following [2] and [1].

Locally, this has the form

$$\begin{aligned} \overset{c}{D}_{\delta_k}\delta_j &= F^i_{jk}\delta_i, & \overset{c}{D}_{\dot{\partial}_k}\delta_j &= C^i_{jk}\delta_i, \\ \overset{c}{D}_{\delta_k}\dot{\partial}_j &= F^i_{jk}\dot{\partial}_i, & \overset{c}{D}_{\dot{\partial}_k}\dot{\partial}_j &= C^i_{jk}\dot{\partial}_i. \end{aligned} \tag{3.1}$$

where

$$F^i_{jk} = F^i_{kj},\ C^i_{jk} = C^i_{kj}, \tag{3.2}$$

and the condition that $\overset{c}{D}$ is metrical with respect to G is equivalent to

$$\begin{aligned} g_{ij|k} &:= \delta_k g_{ij} - F^h_{ik} g_{jh} - F^h_{jk} g_{ih} = 0, \\ g_{ij}|_k &= \dot{\partial}_k g_{ij} - C^h_{ik} g_{jh} - C^h_{jk} g_{ih} = 0. \end{aligned} \tag{3.3}$$

Then, from (3.2) and (3.3), F^i_{jk} and C^i_{jk} are uniquely determined in the form

$$\begin{aligned} F^i_{jk} &= \frac{1}{2} g^{ih}(\delta_j g_{hk} + \delta_k g_{hj} - \delta_h g_{jk}), \\ C^i_{jk} &= \frac{1}{2} g^{ih}(\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{hj} - \dot{\partial}_h g_{jk}) = \frac{1}{2} g^{ih} \dot{\partial}_h g_{jk} \end{aligned} \tag{3.4}$$

Notice that although $\overset{c}{D}$ is a metrical connection, it does not coincide with the Levi–Civita connection of G since it has torsion. Indeed, we have

$$T(\delta_k, \delta_j) = R^i_{jk} \dot{\partial}_i,\ T(\dot{\partial}_k, \delta_j) = C^i_{jk} \delta_i + P^i_{jk} \dot{\partial}_i, \tag{3.5}$$

where $P^i_{jk} = \dot{\partial}_k(N^i_j) - F^i_{kj}$. The d-tensor fields R^i_{jk}, C^i_{jk}, P^i_{jk} vanish only for very particular Lagrangians. For instance, if we consider a Riemannian metric $(\gamma_{ij}(x))$ on$M$ and we put

$$L(x, y) = \gamma_{ij}(x) y^i y^j, \tag{3.6}$$

we obtain a Lagrangian for which $P^i_{jk} = C^i_{jk} = 0$ but $R^i_{jk} \neq 0$ unless if the Riemannian metric $(\gamma_{ij}(x))$ is flat. Now we can prove a simple but important result.

Theorem 3.1. *The linear Cartan connection of the symplectic manifold (TM, Ω_L) is an almost symplectic connection.*

Proof. Using $\Omega_L(X, FY) = G(X, Y)$ and $\overset{c}{D}F = 0$ in the form $\overset{c}{D}_X FY = F\overset{c}{D}_X Y$, one easily obtains

$$(D_X \Omega_L)(Y, Z) = (D_X G)(FY, Z) = 0, \text{ for every } X, Y, Z \in \chi(TM), \text{ q.e.d.}$$

We notice that $\overset{c}{D}$ is completely determined by L. For the Lagrangian (3.6), it reduces to the Levi–Civita connection of the Riemannian metric $(\gamma_{ij}(x))$.

In the proof of Theorem 3.2 we have used the both conditions $\overset{c}{D}F = 0$ and $\overset{c}{D}G = 0$. We may ask if there exists an almost symplectic connection on TMpreserving the horizontal and vertical distributions i.e. a distinguished linear connection satisfying only one or none from these two conditions.

Let D be a distinguished linear connection, shortly a d-connection,on TM. We follow the theory from [2, Ch.3] where such connections are considered on the total space of a vector bundle.

Using the projectors h and v, we have the following decomposition of D:

$$D_XY = hD_{hX}hY + vD_{vX}vY + hD_{vX}hY + vD_{hX}vY,\ X,Y \in \chi(TM). \tag{3.7}$$

When we take

$$hD_{vX}hY = h[vX,hY],\ vD_{hX}vY = v[hX,vY], \tag{3.8}$$

the definition of a connection is respected. The first two terms in the right hand of (3.7) will be determined from the condition $D\Omega_L = 0$, which gives

$$\begin{aligned} &\Omega_L(hD_{hX}hY, hZ) = 0 \\ &\Omega_L(hD_{hX}hY, vZ) = (hX)\Omega_L(hY, vZ) - \Omega_L(hY, v[hX, vZ]), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\Omega_L(vD_{vX}vY, vZ) = 0 \\ &\Omega_L(vD_{vX}vY, hZ) = (vX)\Omega_L(hY, vZ) - \Omega_L(h[hX, vZ], vY). \end{aligned} \tag{3.10}$$

With $hD_{hX}hY$ and $vD_{vX}vY$ uniquely determined from (3.9) and (3.10), respectively, and (3.8), the condition $D\Omega_L = 0$ holds.

The d-connection D is not an almost complex one i.e. $DF \neq 0$ nor a metrical one i.e. $DG \neq 0$.

From (3.7)–(3.10) and $d\Omega_L = 0$ it follows that the torsion of$D$ vanishes if and only if $v[hX,hY] = 0$, that is, the horizontal distribution is integrable. Thus D becomes a symplectic connection only in a very restrictive condition. Now we wish to avoid it and in order to do so we have to renounce to the condition that D is a d-connection. We shall determine a set of symplectic connections for (TM, Ω_L) as a subset of all linear connections on TM and we single out one which is completely determined by L.

Theorem 3.2. *There exists a linear connection ∇ on TM which is almost symplectic with respect to Ω_L, is without torsion and depends on L only.*

Proof. Having the linear Cartan connection $\overset{c}{D}$, we set $\nabla_XY = \overset{c}{D}_XY + A(X,Y)$ for some tensor field A of type $(1,2)$ and $X,Y \in \chi(TM)$. The condition that the torsion of ∇ vanishes reads

$$T(X,Y) + A(X,Y) - A(Y,X) = 0, \tag{3.11}$$

and since $\overset{c}{D}$ is an almost symplectic connection, ∇ is almost symplectic connection if and only if

$$\Omega_L(A(X,Y),Z) + \Omega_L(Y, A(X,Y)) = 0,\ X,Y,Z \in \chi(TM). \tag{3.12}$$

Locally, we put

$$\begin{aligned} &A(\delta_k, \delta_j) = A^i_{jk}\dot\partial_i, \quad A(\dot\partial_k, \delta_j) = E^i_{jk}\dot\partial_i, \\ &A(\delta_k\dot\partial_j) = D^i_{jk}\delta_i, \quad A(\dot\partial_k, \dot\partial_j) = B^i_{jk}\delta_i. \end{aligned} \tag{3.13}$$

Thus we already took a particular form of A. Then (3.11) is equivalent with

$$R^i_{jk} = A^i_{jk} - A^i_{kj},\ E^i_{jk} = -P^i_{kj},\ D^i_{jk} = C^i_{jk},\ B^i_{jk} = B^i_{kj}, \tag{3.14}$$

and (3.12) is equivalent to

$$\begin{aligned} A^h_{ik}g_{hj} - A^h_{jk}g_{hi} = 0, \quad D^h_{ik}g_{hj} - D^h_{jk}g_{hi} = 0, \\ E^h_{ik}g_{hj} - E^h_{jk}g_{hi} = 0, \quad B^h_{ik}g_{hj} - B^h_{jk}g_{hi} = 0. \end{aligned} \tag{3.15}$$

The tensorial eqs. (3.15) could be solved using the Obata operators associated to $(g_{ij}(x, y))$. For brevity, we shall not introduce them here. Instead, we check that the system of eqs. (3.14) and (3.15) has the solution

$$\begin{aligned} &A^i_{jk} = \frac{1}{3}(R^i_{jk} + g^{ih}R_{jhk}),\ E^i_{jk} = -P^i_{kj}, \\ &D^i_{jk} = C^i_{jk},\ B^i_{jk} = \frac{1}{2}(X^i_{jk} + g^{ih}g_{\ell j}X^\ell_{hk}), \end{aligned} \tag{3.16}$$

for some X which satisfies

$$X^i_{jk} = X^i_{kj},\ g_{sj}X^s_{hk} = g_{sk}X^s_{hj},\ \text{otherwise arbitrary.} \tag{3.17}$$

Indeed, the first eq. in (3.14) is verified by virtue of (2.5). The others are clearly verified. In (3.15), the first, the second and the fourth equations become identities by virtue of (3.16). The third is equivalent to $P^h_{ik}g_{hj} = P^h_{jk}g_{hi}$. Inserting P^h_{ik}, after some calculation we find that this equation is equivalent to $g_{ij||k} = g_{ij||k}$, which by (2.5) is an identity. Notice that the condition (2.5), that is $d\Omega_L = 0$ is essential in the solving of (3.14) and of (3.15) as well.

The connection ∇ is determined by $\overset{c}{D}$, the torsion R^i_{jk} and P^i_{jk} as well as by the unknown d-tensor field X^i_{jk} satisfying (3.17). We may take $X^i_{jk} = C^i_{jk}$ since $C_{ijk} = g_{is}C^s_{jk}$ is a completely symmetric d-tensor. This choice singles out a symplectic connection $\overset{s}{\nabla}$ that depends on L only. The theorem is proved.

We remark that the choice which we have made is not unique. Thus $\overset{s}{\nabla}$ is not canonical in any way. However we shall treat only it in the following. And we denote it simply by ∇.

4 Symplectic curvature tensor field of the symplectic connection ∇

In[3], I. Vaisman established the decomposition of the space of tensors which have the symmetries of the curvature of a symplectic connection into $\mathrm{Sp}(n)$–irreducible components. Accordingly, he discovered three classes of symplectic connections: flat, Ricci flat and with reducible curvature. A natural

question is to which class our symplectic connection ∇ belongs. For obtaining an answer we have to compute the symplectic curvature tensor of ∇. We shall do this in the adapted bases $(\delta_i, \dot{\partial}_i)$.

Let ${}^{\nabla}R$ and ${}^{D}R$ be the curvature tensor of type $(1,3)$ of ∇ and D, respectively. Using $\nabla = \overset{c}{D} + A$ we get

(4.1)
$$\begin{aligned}
&{}^{\nabla}R(X,Y)Z = {}^{D}R(X,Y)Z + (D_XA)(Y,Z) - (D_YA)(X,Z) + \\
&+A(T(X,Y),Z) + A(X,A(Y,Z)) - A(Y,A(X,Z)),\ \ X,Y,Z \in \chi(TM).
\end{aligned}$$

where T is the torsion of D locally given by (3.5). With the notations (3.9), the local components of $(D_XA)(Y,Z)$ are given by

$$\begin{aligned}
&(D_{\delta_k}A)(\delta_j,\delta_h) = A^i_{hj|k}\delta_i, && (D_{\delta_k}A)(\delta_j,\dot{\partial}_h) = D^i_{hj|k}\delta_i,\\
&(D_{\delta_k}A)(\dot{\partial}_j,\delta_h) = E^i_{hj|k}\dot{\partial}_i, && (D_{\delta_k}A)(\dot{\partial}_j,\dot{\partial}_h) = B^i_{hj|k}\delta_i,\\
&(D_{\dot{\partial}_k}A)(\delta_j,\dot{\partial}_h) = A^i_{hj}|_k\dot{\partial}_i, && (D_{\dot{\partial}_k}A)(\delta_j,\dot{\partial}_h) = D^i_{hj|k}\delta_i,\\
&(D_{\dot{\partial}_k}A)(\dot{\partial}_j,\delta_h) = E^i_{hj}|_k\dot{\partial}_i, && (D_{\dot{\partial}_k}A)(\dot{\partial}_j,\dot{\partial}_h) = B^i_{hj}|_k\delta_i,
\end{aligned} \tag{4.2}$$

where $(_{|k})$ and $(|k)$ denote the h- and v-covariant derivatives with respect to D. The curvature operator ${}^{D}R(X,Y)$ carries horizontal vector fields to horizontals and the vertical vector fields to verticals. Its action on horizontals is as follows

$$\begin{aligned}
{}^{D}R(\delta_k,\delta_j)\delta_h &= R_h{}^i{}_{jk}\delta_i,\\
{}^{D}R(\dot{\partial}_k,\delta_j)\delta_h &= P_h{}^i{}_{jk}\delta_i,\\
{}^{D}R(\dot{\partial}_k,\dot{\partial}_j)\delta_h &= S_h{}^i{}_{jk}\delta_i,
\end{aligned} \tag{4.3}$$

and its action on verticals is similarly determined by the same d-tensors $R_h{}^i{}_{jk}$, $P_h{}^i{}_{jk}$, $S_h{}^i{}_{jk}$, given by

$$\begin{aligned}
R_h{}^i{}_{jk} &= \delta_k F^i_{hj} + F^s_{hj}F^i_{sk} - (j/k) + C_{hs}R^s_{jk},\\
P_h{}^i{}_{jk} &= \dot{\partial}_k F^i_{hj} - C^i_{hk|j} + C^i_{hs}P^s_{jk},\\
S_h{}^i{}_{jk} &= \dot{\partial}_k C^i_{hj} + C^s_{hj}C^i_{sk} - (j/k),
\end{aligned} \tag{4.4}$$

where (j/k) means the preceding terms with k changed to j and j changed to k.

The curvature operator ${}^{\nabla}R(X,Y)$ does not preserve the horizontal and vertical distributions. As such ${}^{\nabla}R$ has twelve components. We put

$$\begin{aligned}
{}^{\nabla}R(\delta_k,\delta_j)\delta_h &= {}^{\nabla}R_h{}^i{}_{jk}\delta_i + K_h{}^i{}_{jk}\dot{\partial}_i,\\
{}^{\nabla}R(\dot{\partial}_k,\delta_j)\delta_h &= {}^{\nabla}P_h{}^i{}_{jk}\delta_i + K_h{}^i{}_{jk}\dot{\partial}_i,\\
{}^{\nabla}R(\dot{\partial}_k,\dot{\partial}_j)\delta_h &= {}^{\nabla}S_h{}^i{}_{jk}\delta_i + M_h{}^i{}_{jk}\dot{\partial}_i,\\
{}^{\nabla}R(\delta_k,\delta_j)\dot{\partial}_h &= \widetilde{K}_h{}^i{}_{jk}\delta_i + \widetilde{R}_h{}^i{}_{jk}\dot{\partial}_i,\\
{}^{\nabla}R(\dot{\partial}_k,\delta_j)\dot{\partial}_h &= \widetilde{L}_h{}^i{}_{jk}\delta_i + \widetilde{P}_h{}^i{}_{jk}\dot{\partial}_i,\\
{}^{\nabla}R(\dot{\partial}_k,\dot{\partial}_j)\dot{\partial}_h &= \widetilde{M}_h{}^i{}_{jk}\delta_i + \widetilde{S}_h{}^i{}_{jk}\dot{\partial}_i.
\end{aligned} \tag{4.5}$$

Using (3.5) and (4.2) an explicit form of these components is obtained as follows

$$
\begin{aligned}
{}^{\nabla}R_h{}^i{}_{jk} &= R_h{}^i{}_{jk} + (A^s_{hj}D^i_{sk} - (k/j)),\\
K_h{}^i{}_{jk} &= A^i_{hj|k} - (k/j) + R^s_{jk}E^i_{hs}.
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
{}^{\nabla}P_h{}^i{}_{jk} &= P_h{}^i{}_{jk} - E^s_{hk}D^i_{sj} + A^s_{hj}B^i_{sk},\\
L_h{}^i{}_{jk} &= A^i_{hj|k} - E^i_{hk|j} + C^s_{jk}A^i_{hs} + P^s_{jk}E^i_{hs}.
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
{}^{\nabla}S_h{}^i{}_{jk} &= S_h{}^i{}_{jk} + (E^s_{hj}B^i_{sk} - (k/j)),\\
M_h{}^i{}_{jk} &= E^i_{hj}\big|k - (k/j).
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
\widetilde{K}_h{}^i{}_{jk} &= D^ihj_{|k} - (j/k) + R^s_{jk}B^i_{hs},\\
\widetilde{R}_h{}^i{}_{jk} &= R_h{}^i{}_{jk} + (D^s_{hj}A^i_{sk} - (k/j)).
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\widetilde{L}_h{}^i{}_{jk} &= D^ihj\big|_k + C^s_{jk}D^i_{hs} + P^s_{jk}B^i_{hs},\\
\widetilde{P}_h{}^i{}_{jk} &= P_h{}^i{}_{jk} + D^s_{hj}E^i_{sk} - B^s_{hk}A^i_{sj}.
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\widetilde{M}_h{}^i{}_{jk} &= B^ihj\big|k - (j/k)\\
\widetilde{S}_h{}^i{}_{jk} &= S_h{}^i{}_{jk} + (B^s_{hj}E^i_{sk} - (k/j)).
\end{aligned}
\tag{4.11}
$$

Then, if we take in (4.6)–(4.11), $A^i_{jk} = \dfrac{1}{3}(R^i_{jk} + g^{ih}R_{jhk})$, $E^i_{jk} = -P^i_{jk}$ and $D^i_{jk} = B^i_{jk} = C^i_{jk}$, we obtain the twelve components of the curvature tensor of type $(1,3)$ of ∇.

The symplectic curvature tensor is defined by

$$
S(X_2, X_2, X_3, X_4) = \Omega_L({}^{\nabla}R(X_3, X_4)X_2, X_1),\ \ X_1, ...X_4 \in \chi(TM).
\tag{4.12}
$$

This is skew symmetric in the last two arguments and symmetric with respect to the first two arguments. Moreover, the cyclic sum over the last three arguments vanishes, [3]. It is locally determined by the twelve tensors ${}^{\nabla}R_{hijk}, ..., \widetilde{S}_{hijk}$ from (4.6)–(4.11) with the upper index brought down with (g^{hs}) on the second place.

The Ricci curvature tensor of ∇ is defined by the usual formula

$$
\sigma(X, Y) = \mathrm{Tr}(V \to {}^{\nabla}R(V, X)Y).
\tag{4.13}
$$

A direct calculation gives

$$
\begin{aligned}
\sigma(\delta_j, \delta_h) &= {}^{\nabla}R_h{}^i{}_{jki} + L_h{}^i{}_{ji},\\
\sigma(\delta_j, \dot{\partial}_h) &= \widetilde{K}_h{}^i{}_{ji} + \widetilde{P}_h{}^i{}_{ji},\\
\sigma(\dot{\partial}_j, \dot{\partial}_h) &= -\widetilde{L}_k{}^i{}_{ji} + \widetilde{S}_h{}^i{}_{ji}.
\end{aligned}
\tag{4.14}
$$

The components of curvature tensors just found are quite complicated. Thus it is quite sure that for a general Lagrangian L, the symplectic connection ∇ is a general one, too. The above formula simplify for the Lagrangian defined by a Riemannian metric $\gamma_{ij}(x)$ as in (3.6).

Let $\gamma^i_{jk}(x)$ be the Christoffel symbols of $\gamma_{ij}(x)$ and $r_h{}^i{}_{jk}(x)$ its curvature tensor. Then $\Omega_L = \gamma_{ij}(x)\delta y^i \wedge dx^j$ and ∇ takes the form

$$\begin{aligned} \nabla_{\delta_k}\delta_j &= \gamma^i_{jk}\delta_i + A^i_{jk}\dot{\partial}_i,\ \nabla_{\dot{\partial}_k}\delta_j = 0, \\ \nabla_{\delta_k}\dot{\partial}_j &= \gamma^i_{jk}\dot{\partial}_i,\ \nabla_{\dot{\partial}_k}\dot{\partial}_j = 0, \end{aligned} \tag{4.15}$$

where A^i_{jk} is given by (3.12) with $R^i_{jk} = r_h{}^i{}_{jk}(x)y^h$. An inspection of (4.4), (4.6)–(4.11) shows that the nonzero components of ${}^\nabla R$ are the following ones

$$ {}^\nabla R_h{}^i{}_{jk} = r_j{}^i{}_{hk}(x), \tag{4.16}$$

$$K_h{}^i{}_{jk} = A^i_{hj|k} - (k/j) = \frac{1}{3}(r_q{}^i{}_{hj;k} + r_j{}^i{}_{hq;k})y^q - (k/j), \tag{4.17}$$

$$L_h{}^i{}_{jk} = A^i_{hj}|_k = \frac{1}{3}(r_k{}^i{}_{hj} + r_j{}^i{}_{hk}), \tag{4.18}$$

where $(;k)$ means the covariant derivative with respect to the Levi–Civita connection of (γ_{ij}). From (4.16)–(4.18) it follows

Theorem 4.1. *Let be the symplectic manifold (TM, Ω_L) for $L(x,y) = \gamma_{ij}(x)y^iy^j$ and $(\gamma_{ij}(x))$ a Riemannian metric. The symplectic connection ∇ is flat if and only if (γ_{ij}) is flat.*

From (4.14) it results that the only non–zero component of the Ricci curvature tensor of ∇ is

$$\sigma(\delta_j, \delta_h) = \frac{2}{3} r_{jh}(x), \tag{4.19}$$

where $r_{jh}(x) = r_j{}^i{}_{hi}(x)$ is the Ricci tensor of $\gamma_{ij}(x)$. Thus we have

Theorem 4.2. *The same hypothesis as in Theorem* 4.1. *The symplectic connection ∇ is Ricci flat if and only if $\gamma_{ij}(x)$ is Ricci flat.*

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METRIZABLE LINEAR CONNECTIONS IN VECTOR BUNDLES

BY

M. ANASTASIEI

Dedicated to Professor Dr. Lajos Tamássy at his 80th anniversary

Abstract

A linear connection ∇ in a vector bundle is said to be metrizable if the vector bundle admits a Riemannian metric h with the property $\nabla h = 0$. Sufficient conditions for the linear connection ∇ to be metrizable are provided.

Mathematics Subject Classification: Primary 53C60; Secondary 53C05.
Key words and phrases: vector bundles, linear connections, metrizability

Introduction

The problem of the metrizability of a linear connection was treated by many authors in various contexts (see the paper [7] by L. Tamassy and the references therein). When a linear connection ∇ in a vector bundle $\xi = (E, p, M)$ is metrizable, its parallel translations are isometries with respect to any Riemannian metric h in ξ with $\nabla h = 0$. Using a local chart around a point x in M, the holonomy group $\phi(x)$ may be identifed with a subgroup of $GL(m, \mathbb{R})$, where m is the dimension of fibre. With this identification, a necessary condition for ∇ be metrizable is that the holonomy group to be contained in the orthogonal group $O(m)$. We prove two versions of the converse of this fact (Theorems 3.1 and 3.2). Then, we are dealing with the same problem when the vector bundle ξ is endowed with a Finsler function. The linear connection ∇ induces a nonlinear connection on E and a linear connection D in the vertical vector bundle over E. The Finsler function F defines a Riemannian metric g in the vertical vector bundle over E. We show that if g is covariant constant on horizontal directions, then ∇ is metrizable (Theorem 4.2). When the tangent bundle of a manifold M is endowed with a Finsler function F one says that (M, F) is a Finsler manifold. In this case our result has to be compared with the one due to Z. Szabó, ([6]) regarding the metrizability of the Berwald connection.

If the cotangent bundle of a manifold M is endowed with a Finsler function K, then the pair (M, K) is called a *Cartan space*. This notion was introduced and studied by R. Miron in [3]. In this case Theorem 4.1 has to be compared with our previous results on the metrizability of Berwald–Cartan connection [1].

The first two sections of the paper are devoted to some preliminaries from the theory of vector bundles and linear connections in vector bundles.

1 Vector bundles

Let $\xi = (E, p, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n + m$ and $p : E \to M$ is a smooth submersion. The fibres $E_x = p^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be an atlas on M. A vector bundle atlas is $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}_{\alpha\in A}$ with the bijections $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha = (p(u), \varphi_{\alpha,p(u)}(u))$, where $\varphi_{\alpha,p(u)} : E_p(u) \to \mathbb{R}^m$ is a bijection. The given atlas on M and a vector bundle atlas provide an atlas $(p^{-1}(U_\alpha), \phi_\alpha)_{\alpha\in A}$ on E. Here $\phi_\alpha : p^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\phi_\alpha(u) = (\psi_\alpha(p(u)), \varphi_{\alpha,p(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^m$ and we take (x^i, y^a) as local coordinates on E. If (U_β, ψ_β) is such that $x \in U_\alpha \cap U_\beta \neq \emptyset$ and $\psi_\beta(x) = (\widetilde{x}^i)$, then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n),\ \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \tag{1.1}$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi^{-1}_{\alpha,x}(e_a) = \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ takes the form $u = y^a \varepsilon_a(x)$. We put $\widetilde{y}^a = M^a_b(x)y^b$ with $\operatorname{rank}(M^a_b(x)) = m$. Then $\phi_\beta \circ \phi_\alpha^{-1}$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \quad \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{y}^a = M^a_b(x)y^b, \qquad\quad \operatorname{rank}(M^a_b(x)) = m. \end{aligned} \tag{1.2}$$

The indices $i, j, k, ..., a, b, c, ..$ will take the values $1, 2, ..., n$ and $1, 2, ..., m$, respectively. The Einstein convention on summation will be used.

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E, respectively and by $\mathcal{X}(M)$, resp. $\Gamma(E)$, $\mathcal{X}(E)$ the module of sections of the tangent bundle of M, resp. of the bundle ξ and of the tangent bundle of E. On U_α, the vector fields $\left(\partial_k := \dfrac{\partial}{\partial x^k}\right)$ provide a local basis for $\mathcal{X}(U_\alpha)$. The sections $\varepsilon_a : U_\alpha \to p^{-1}(U_\alpha)$ given by $\varepsilon_a(x) = \varphi^{-1}_{\alpha,x}(e_a)$ will be taken as canonical basis for $\Gamma(p^{-1}(U_\alpha))$ and a section $A : U_\alpha \to p^{-1}(U_\alpha)$ will take the form $A(x) = A^a(x)\varepsilon_a(x)$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ. We take as local basis of $\Gamma(E^*)$ on U_α, the sections $\theta^a : U_\alpha \to p^{*-1}(U_\alpha)$, $x \to \theta^a(x) \in E^*_x$ such that $\theta^a(\varepsilon_b(x)) = \delta^a_b$.

Next, we may consider the tensor bundle of type $(r,s)\mathcal{T}^r_s(E) := E\underbrace{\otimes\cdots\otimes}_{r}E$ $\otimes E^*\underbrace{\otimes\cdots\otimes}_{s}E^*$ over M and its sections. For $g\in\Gamma(E^*\otimes E^*)$ we have the local representation $g = g_{ab}(x)\theta^a\otimes\theta^b$. As $E^*\otimes E^*\cong L_2(E,\mathbb{R})$, we may regard g as a smooth mapping $x\to g(x): E_x\times E_x\to\mathbb{R}$ with $g(x)$ a bilinear mapping given by $g(x)(s_a,s_b)=g_{ab}(x)$.

If the mapping $g(x)$ is symmetric i.e. $g_{ab}=g_{ba}$ and positive–definite i.e. $g_{ab}(x)\zeta^a\zeta^b>0$ for every $0\neq(\zeta^a)\in\mathbb{R}^m$, one says that g defines a Riemannian metric in the vector bundle ξ.

The sets of sections $\Gamma(T^r_s(E))$ are $\mathcal{F}(M)$-modules for every natural numbers r,s. On the sum $\bigoplus_{r,s}\Gamma(T^r_s(E))$ a tensor product can be defined and one gets a tensorial algebra $\mathcal{T}(E)$. For the vector bundle (TM,τ,M) this reduces to tensorial algebra of the manifold M.

2 Linear connections in a vector bundle

Definition 2.1. A linear connection in the vector bundle $\xi=(E,p,M)$ is a mapping $\nabla:\mathcal{X}(M)\times\Gamma(E)\to\Gamma(E)$, $(X,A)\to\nabla_X A$ which is $\mathcal{F}(M)$-linear in the first argument, additive in the second and

$$\nabla_X(fA)=X(f)A+f\nabla_X A,\ \ f\in\mathcal{F}(M). \tag{2.1}$$

For $X=X^k(x)\partial_k$ and $A=A^a(x)\varepsilon_a(x)$, we get

$$\nabla_X A = X^k(\partial_k A^a+\Gamma^a_{bk}(x)A^b)\varepsilon_a(x), \tag{2.2}$$

where the local coefficients $\Gamma^a_{bk}(x)$ are defined by

$$\nabla_{\partial_k}\varepsilon_b=\Gamma^a_{bk}\varepsilon_a. \tag{2.3}$$

If $\widetilde{\Gamma}^c_{dj}$ are the local coefficients of ∇ on U_β such that $U_\alpha\cap U_\beta\neq\emptyset$, then we have

$$\widetilde{\Gamma}^c_{dj}(\widetilde{x}(x)) = M^c_a(x)(M^{-1})^b_d\frac{\partial x^k}{\partial\widetilde{x}^j}\Gamma^a_{bk}(x)-\frac{\partial M^c_b}{\partial x^k}\,\frac{\partial x^k}{\partial\widetilde{x}^j}(M^{-1})^b_d. \tag{2.4}$$

A section A of ξ is called *parallel* if $\nabla_X A=0$ for every $X\in\mathcal{X}(M)$.

The linear connection ∇ induces operators of covariant derivative ∇_k in the tensorial algebra $\mathcal{T}(E)$ taking $\nabla_k f=\partial_k f$, $\nabla_k\beta_a=\partial_k\beta_a-\Gamma^c_{ak}\beta_c$ and requiring that ∇_k to satisfy the Newton–Leibniz rule with respect to the tensor product and to commute with the all contractions.

Let $c:[0,1]\to M$ be a curve on M and $A: t\to A(t):=A(c(t))$ a section of ξ along the curve c. Then $\nabla_{\dot c(t)}A =: \dfrac{\nabla A}{dt}$ is called the covariant derivative of A along c.

On $U_\alpha \cap c[0,1]$ if one puts $c(t) = (x^i(t))$, we get

$$\frac{\nabla A}{dt} = \left(\frac{dA^a}{dt} + \Gamma^a_{bk}(x(t))A^b\frac{dx^k}{dt}\right)\varepsilon_a. \tag{2.5}$$

The section $t \to A(t)$ is said to be *parallel* on c if $\dfrac{\nabla A}{dt} = 0$. This means that the functions $(A^a(t))$ have to be solutions of the following system of ordinary linear differential equations

$$\frac{dA^a}{dt} + \Gamma^a_{bk}(x)A^b\frac{dx^k}{dt} = 0. \tag{2.6}$$

For given initial conditions $A^a(0) = (u^a) \in E_{c(0)}$ the system (2.6) admits a unique solution that can be prolonged beyond U_α providing a parallel section A along c. If we associates to $(u^a) = A^a(0)$ the element $(v^a) = A^a(1) \in E_{c(1)}$ one gets a linear isomorphism $P_c : E_{c(0)} \to E_{c(1)}$, called the *parallel translation* of $E_{c(0)}$ to $E_{c(1)}$ along c. The parallel translations can be defined along any curve or segment of curve providing linear isomorphisms between fibres in various point of curves on M. In particular, if one considers the loops with the origin in $x \in M$, the corresponding parallel translations as linear isomorphisms $E_x \to E_x$ can be composed and a group $\phi(x)$ called the holonomy group in $x \in M$ is obtained.

When M is connected, the holonomy groups $\phi(x)$, $x \in M$, are isomorphic and one speaks about the holonomy group ϕ associated to or defined by ∇.

The covariant derivative along c can be recovered from parallel translations according to the following known

Lemma 2.1. *Let A be a section of ξ along a curve on M, c: $t \to c(t)$, $t \in \mathbb{R}$, starting from $x = c(0)$. Then*

$$(\nabla_{\dot{c}(0)}A)(x) = \lim_{t\to 0}\frac{1}{t}(P_c(A(t)) - A(0)), \tag{2.7}$$

where $P_c : E_{c(t)} \to E_x$ is the parallel translation along c.

3 A sufficient condition for ∇ be metrizable

Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$. Assume that the manifold M is connected. One says that ∇ is *metrizable* if there exists a Riemannian metric g in ξ such that $\nabla g = 0$. When ∇ is metrizable all parallel translations $P_c : (E_x, g_x) \to (E_y, g_y)$ for any curve c and any points x, y joining them in M are isometries. In particular, the holonomy group $\phi(x)$ is a subgroup of the orthogonal group of (E_x, g_x). These facts follow from

Lemma 3.1. *Let g be any Riemannian metric in the vector bundle ξ and $c : t \to c(t)$, $t \in \mathbb{R}$, a curve in M with $c(0) = x$. Then*

$$\left(\nabla_{\dot{c}(0)}g\right)(A, B) = \lim_{t\to 0}\frac{1}{t}(g_{c(t)}(P_cA, P_cB) - g_x(A, B)), \tag{3.1}$$

where $A, B \in E_x$ and $P_c : E_x \to E_{c(t)}$ is the parallel translation along c.

Proof. Let $\widetilde{A}, \widetilde{B}$ be sections of ξ which are parallel on c such that $\widetilde{A}(0) = A$, $\widetilde{B}(0) = B$. Then $P_c A = \widetilde{A}(t)$ and $P_c(B) = \widetilde{B}(t)$. By the Taylor theorem and using the condition that $\widetilde{A}$ and $\widetilde{B}$ are parallel sections on c, in the natural basis (ε_a) we get $(P_c A)^a = \widetilde{A}^a(t) = A^a + \dfrac{d\widetilde{A}}{dt}(\tau)t = A^a - \Gamma^a_{ck}(x(\tau))\widetilde{A}^c(\tau)\dfrac{dx^k}{dt}t$ and a similar formula for $(P_c B)^b$, $a, b = 1, 2, ..., m$. Then, using again the Taylor theorem, omitting the terms which contain t^2, we may write:

$$g_{ab}(t)(P_cA)^a(P_cB)^b - g_{ab}(x)A^aB^b =$$

$$\left(g_{ab}(x) + \frac{dg_{ab}}{dt}(\theta)t\right)(P_cA)^a(P_cB)^b - \tag{3.2}$$

$$-g_{ab}(x)A^aB^b = \left(\frac{dg_{ab}}{dt} - g_{ac}\Gamma^c_{bk}\frac{dx^k}{dt} - g_{cb}\Gamma^c_{ak}\frac{dx^k}{dt}\right)t,$$

where the terms in the last paranthesis are computed for $\tau, \tau', \theta \in (0, t)$.

Dividing in (3.2) by t and taking $t \to 0$, one obtains (3.1).

By Lemma 3.1 we have also that if all parallel translations of ∇ are isometries with respect to g, then $\nabla g = 0$. Thus, in order to prove that ∇ is metrizable we need to find a Riemannian metric g such that all parallel translations of ∇ to be isometries with respect to g. Taking an arbitrary bundle chart $(U_\alpha, \varphi_\alpha, \mathbb{R}^m)$, using the linear isomorphism $\varphi_{\alpha,x} : E_x \to \mathbb{R}^m$, we may identify $\phi(x)$, $x \in U_\alpha$, with a subgroup of $GL(\mathbb{R}^m)$. When ∇ is metrizable, by Lemma 3.1 it follows that this subgroup is contained in the orthogonal group $O(m)$. Therefore, a necessary condition for ∇ be metrizable is that its holonomy group to be contained in $O(m)$. We show two versions of the converse.

Theorem 3.1. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Assume that there exists a point $x_0 \in M$ such that the holonomy group $\phi(x_0)$ is contained in the orthogonal group of E_{x_0} when E_{x_0} is regarded as being isomorphic with the Euclidean space $(R^m, <, >)$ via a fixed bundle chart. Then ∇ is metrizable.*

Proof. Let h_0 be the inner product on E_{x_0} induced by $<, >$ via the bundle chart $(U_\alpha, \varphi_\alpha, \mathbb{R}^m)$, $x_0 \in U_\alpha$, that is,

$$h_0(u, v) = <\varphi_{\alpha,x_0}u, \varphi_{\alpha,x_0}v> . \tag{$*$}$$

By hypothesis this inner product is invariant under the group $\phi(x_0)$. Let be any $x \in M$. We join x with x_0 using a curve $c : [0, 1] \to M$, $c(0) = x$, $c(1) = x_0$, consider the parallel translation $P_c : E_x \to E_{x_0}$ and define an inner product h_x in E_x by

$$h_x(A, B) = h_0(P_cA, P_cB),\ A, B \in E_x. \tag{3.3}$$

Lemma 3.2. *The inner product h_x does not depend on the curve c.*

Indeed, if $\widetilde{c}$ is another curve joining x with x_0, we consider the reverse c_- of c and the loop $\widetilde{c} \circ c_-$ in x_0. It follows that $h_0\left(P_{\widetilde{c}\circ c_-}u, P_{\widetilde{c}\circ c_-}v\right) = h_0(u,v)$, $u, v \in E_{x_0}$. Inserting here $u = P_c A$ and $v = P_c B$ and taking into account (3.3), the Lemma follows.

The mapping $x \to h_x$ is smooth since P_c smoothly depends on x according to the general theory of differential equations. Thus we obtain a Riemannian metric h in ξ. The parallel translations of ∇ are isometrics with respect to h. Indeed, for y a point of M different from x, any parallel translation from E_x to E_y has the form $P_{\sigma_- \circ c} = P_{\sigma_-} \circ P_c$ for σ_- the reverse of a curve σ joining y with x_0. This is an isometry as a product of isometries. Therefore, we may conclude using Lemma 3.1, that $\nabla h = 0$. q.e.d.

The following version of the Theorem 3.1 extends to the vector bundle setting a result of B.G. Schmidt [5].

Theorem 3.2. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Assume that for a fixed $x_0 \in M$, the holonomy group $\phi(x_0)$ leaves invariant a given positive–definite quadratic form h_0 on E_{x_0}. Then there exists a Riemannian metric h in ξ such that $\nabla h = 0$.*

Proof. Let denote by the same letter h_0 the inner product in E_{x_0} defined by the quadratic form h_0. This inner product could be obtained by transferring one from $\mathbb{R}^m$ using a bundle chart. By hypothesis the inner product h_0 is invariant under $\phi(x_0)$. From now on the reasoning proving Theorem 3.1 can be entirely repeated in order to find h such that $\nabla h = 0$.

Remark 3.1. The Riemannian metric h found in Theorem 3.1 is not unique and is not canonical in any way. The same applies for h found in Theorem 3.2.

4 Another condition for ∇ be metrizable

We are dealing with the problem of the metrizability of a linear connection ∇ in a vector bundle endowed with a Finsler function.

Definition 4.1. Let $\xi = (E, p, M)$ be a vector bundle of rank m. A *Finsler function* on E is a nonnegative real function F on E with the properties

1) F is smooth on $E \setminus \{(x, 0), x \in M\}$,

2) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,

3) The matrix with the entries $g_{ab}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^a \partial y^b}$ is positive definite.

On the manifold E we have the vertical distribution $u \to V_u E = \ker p_{*,u}$ where p_* denotes the differential of p. This is spanned by $\dot{\partial}_a := \dfrac{\partial}{\partial y^a}$. A distribution $u \to H_u E$ which is supplementary to the vertical distribution is called a *horizontal* distribution or a *nonlinear connection* on E. This is usually taken as spanned by $\delta_i = \partial_i - N_i^a(x, y)\dot{\partial}_a$, where the functions

$(N_i^a(x,y))$ are called the *coefficients* of the given nonlinear connection. Under a change of coordinates they behave as follows:

$$(4.1) \qquad \widetilde{N}_j^a \frac{\partial \widetilde{x}^j}{\partial x^k} = M_b^a(x) N_k^b(x,y) - \frac{\partial M_b^a}{\partial x^k} y^b,$$

a fact which is equivalent with

$$(4.1)' \qquad \delta_i = \frac{\partial \widetilde{x}^k}{\partial x^i} \widetilde{\delta}_k.$$

Introducing the horizontal distribution we have

$$(4.2) \qquad T_u E = H_u E \oplus V_u E, \ u \in E.$$

It is convenient to decompose the geometrical objects on E according to (4.2) using the adapted basis $(\delta_i, \dot{\partial}_a)$ and its dual $(dx^i, \delta y^a = dy^a + N_i^a(x,y)dx^i)$.

The linear connection ∇ in ξ defines a nonlinear connection on E taking $N_i^a(x,y) = \Gamma_{bi}^a(x) y^b$. Indeed, using (2.4) it is easy to check that these functions satisfy (4.1). From now on we shall use only the decomposition (4.2) provided by these functions.

Furthermore, the linear connection ∇ induces a linear connection D in the vertical bundle over E as follows: $D : \mathcal{X}(E) \times \Gamma(VE) \to \Gamma(VE)$, $(X,Z) \to D_X Z$ is given for $Z = Z^a \dot{\partial}_a$ by

$$(4.3) \qquad D_{\delta_k} \dot{\partial}_a = \Gamma_{bk}^a(x) \dot{\partial}_a, \ D_{\dot{\partial}_b} \dot{\partial}_a = 0.$$

We call D the vertical lift of ∇ and we use D_{δ_k} for defining a *horizontal* covariant derivative operator in the tensor algebra of the vertical bundle, denoted by $_{|k}$, setting

$$(4.4) \qquad \begin{aligned} & f_{|k} = \delta_k f \text{ for any function on } E \\ & X^a_{|k} = \delta_k X^a + \Gamma_{bk}^a(x) X^b. \end{aligned}$$

For a fixed $x \in E$, the pair (E_x, F_x) is a Minkowski space. Here F_x denotes the restriction of F to E_x and it is obvious that this is a Minkowski norm on E_x.

Now we show that under certain conditions the parallel translations of ∇ are isometries of Minkowski spaces.

Theorem 4.1. *Let $\xi = (E, p, M)$ be a vector bundle of rank m with M connected, endowed with a Finsler function F and with a linear connection ∇ as well. Let $_{|k}$ be the horizontal covariant derivative operator defined by the vertical lift D of ∇. If $F_{|k} = 0$, then the parallel translation defined by ∇, $P_c : (E_x, F_x) \to (E_y, F_y)$ is an isometry of Minkowski spaces for any points $x, y \in M$ and any curve $c : [0,1] \to M$ joining them.*

Proof. Let be $u \in E_x$ and $t \to A(t)$, $t \in [0,1]$ a section of ξ which is parallel along c and $A(0) = u$. Its local components A^a are solutions of the system of differential equations (2.6). And $P_c(u) = A(1) := v$.

We know already that P_c is a linear isomorphism. Let us write the condition $F_{|k} = 0$ for the points $(x(t), A(t))$ of E where $t \to x(t)$ is the local representation of the curve c. We obtain:

$$0 = \left(\frac{\partial F}{\partial x^k} - A^b \Gamma^a_{bk} \frac{\partial F}{\partial y^a} \right) \frac{dx^k}{dt} \overset{(2.6)}{=\!=} \frac{\partial F}{\partial x^k} \, \frac{dx^k}{dt} + \frac{\partial F}{\partial y^a} \, \frac{dA^a}{dt} = \frac{dF(x(t), A(t))}{dt}.$$

Thus the function $F(x(t), A(t))$ is constant. It follows $F(x, u) = F(y, P_c u)$, that is, $F_x(u) = F_y(P_c u)$. In other words, P_c is an isometry of Minkowski spaces (E_x, F_x) and (E_y, F_y). q.e.d.

Corollary 4.1. *In the hypothesis of Theorem* 4.1*, the holonomy group $\phi(x)$ consists of isometries of the Minkowski space (E_x, F_x).*

The functions $g_{ab}(x, y)$ define a Riemannian metric in the vertical bundle over E by $g = g_{ab}(x, y)\delta y^a \otimes \delta y^b$. We call $(g_{ab}(x, y))$ the Finsler metric associated with F.

The condition $F_{|k} = 0$ from the hypothesis of Theorem 4.1 can be replaced with $g_{ab|k} = 0$, because of

Lemma 4.1. $F_{|k} = 0$ *is equivalent with* $g_{ab|k} = 0$.

Proof. The homogeneity of F implies $F^2(x, y) = g_{ab}(x, y)y^a y^b$. Then $F^2_{|k} = 2FF_{|k} = g_{ab|k}y^a y^b + 2g_{ab}y^a_{|k}y^b = g_{ab|k}y^a y^b$ since $y^a_{|k} = 0$. Thus if $g_{ab|k} = 0$, then $F_{|k} = 0$. In order to prove the converse, we notice that $\dot{\partial}_a(H_{|k}) = (\dot{\partial}_a H)_{|k}$ for any function H on E. This follows by a direct calculation taking care that $\dot{\partial}_a H$ is a vertical 1-form. Using this "commutation" formula we get $g_{ab|k} = \dfrac{1}{2}\dot{\partial}_a\dot{\partial}_b(F^2_{|k}) = \dot{\partial}_a\dot{\partial}_b(FF_{|k}) = 0$. q.e.d.

Now we are ready to prove the main result of this section.

Theorem 4.2. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Suppose that E is endowed with a Finsler function F with the associated Finsler metric $g_{ab}(x, y)$. Let $_{|k}$ be the h-covariant derivative operator induced by ∇. If $g_{ab|k} = 0$, then ∇ is metrizable.*

Proof. For a fixed $x_0 \in M$ we have the Minkowski space (E_{x_0}, F_{x_0}). Let G be the group of all linear isomorphisms of E_{x_0} which preserve the set $S_{x_0} = \{u \in E_{x_0}, F_{x_0}(u) = 1\}$. This G is a compact Lie group since S_{x_0} is compact. In our hypothesis, according to Lemma 4.1 and Corollary 4.1, the holonomy group $\phi(x_0)$ is a Lie subgroup of G. Let $<,>$ be any inner product on E_{x_0}. Define a new inner product on E_{x_0} by

$$h_{x_0}(u, v) = \frac{1}{\mathrm{vol}(G)} \int_G < gu, gv > \mu_G, \tag{4.5}$$

for $u, v \in E_{x_0},\ g \in G$ and μ_G the bi-invariant Haar measure on G.

It follows that for every $a \in G$ we have

$$h_{x_0}(au, av) = h_{x_0}(u, v),\ u, v \in E_{x_0}. \tag{4.6}$$

In particular, (4.6) holds for any element of $\phi(x_0) \subset G$. Thus $\phi(x_0)$ leaves invariant the inner product h_{x_0} in E_{x_0}. The inner product h_{x_0} is extended by parallel translations to a Riemannian metric h in ξ. Furthermore, this metric verifies $\nabla h = 0$ since all parallel translations of ∇ become isometries with respect to h. Thus ∇ is metrizable. q.e.d.

Remark 4.1. The Riemannian metric h is not unique and it is not canonical in any way.

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GEOMETRY OF BERWALD VECTOR BUNDLES

by Mihai ANASTASIEI

Dedicated to Prof. Dr. Radu MIRON on the occasion of his 75th birthday

Abstract

Let ξ be a vector bundle endowed with a nonlinear connection N. It is called a Berwald vector bundle if the local coefficients of the Berwald linear connection defined by N do not depend on the variables y in fibres of ξ. Thus they define a linear connection ∇ in ξ. One endows ξ with a regular Lagrangian L. A compatibility condition between L and N is introduced and consequences of it on the holonomy group of ∇ are derived. Assuming that L is homogeneous of degree two in y, one proves that ∇ is metrizable. Some particular cases and examples are discussed.
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Introduction

In the very influential paper [8], R. Miron develops the geometry of the total space of a vector bundle using ideas and techniques from Finsler geometry. He considers on the total space E of the vector bundle $\xi = (E, p, M)$ a distribution that is supplementary to the vertical distribution i.e. a nonlinear connection and decomposes all geometrical objects on E with respect to these distributions. On this way he proposes an elegant treatment of the linear connections and of metrical structures on E. From a nonlinear connection N a linear connection in the vertical bundle over E is easily derived. This is called the Berwald connection associated to N. When it happens that the local coefficients of this connection do not depend on the variables y in fibres, they define a linear connection ∇ in the vector bundle ξ and the pair (ξ, N) will be called a Berwald bundle. Some properties of the pairs (ξ, N) are given in Section 1. Then, in Section 2, we endow E with a regular Lagrangian L and introduce a natural condition of compatibility between N and L. Some direct consequences of this compatibility are given in Proposition 2.1. Then we consider the parallel translations defined by ∇ and we show in Theorem 2.1 that these are compatible with the structures induced by the Lagrangian on the fibres of ξ. In particular, the holonomy group $\phi(x)$, $x \in M$, of ∇

preserves the indicatrix defined by L. The differentials of the elements of the holonomy group $\phi(x)$, provide a group of linear isomorphism of the vertical subspace V_uE, $p(u) = x$. We show in Theorem 2.2 that the elements of this group are also isometries with respect to the pseudo–Riemannian metric induced by L in the vertical bundle over E. In Section 3 we treat the case $L = F^2$, where F is a Finsler function. In this case we prove that ∇ is metrizable, that is there exists a Riemannian metric h in ξ such that $\nabla h = 0$. Some particular cases and examples are discussed in Section 4. The notations and terminology are those from [9] and [5].

1 Berwald vector bundles

Let $\xi = (E, p, M)$, $p : E \to M$, be a vector bundle of rank m. Here M is a smooth i.e. C^∞ manifold of dimension n. The type fibre is $\mathbb{R}^m$ and E is a smooth manifold of dimension $n + m$. The projection p is a smooth submersion. Let $(U, (x^i))$ be a local chart on M and let $\varepsilon_a(x)$, $x \in U$, be a field of local sections of ξ over U. Then every section A of ξ over U takes the form $A = A^a(x)\varepsilon_a(x)$, $x \in U$, and an element $u \in p^{-1}(x) := E_x$ can be written as $u = y^a\varepsilon_a(x)$, $(y^a) \in \mathbb{R}^m$. The indices $i, j, k, ...$ will range over $\{1, 2, ..., n\}$ and the indices $a, b, c, ...$ will take their values in $\{1, 2, ..., m\}$. The convention on summation over repeated indices of the same kind will be used.

The local coordinates on $p^{-1}(U)$ will be (x^i, y^a) and a change of coordinates $(x^i, y^a) \to (\widetilde{x}^i, \widetilde{y}^a)$ on $U \cap \widetilde{U} \neq \emptyset$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n),\ \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n, \\ &\widetilde{y}^a = M^a_b(x)y^b,\ \operatorname{rank}(M^a_b(x)) = m, \quad \forall x \in U \cap \widetilde{U}. \end{aligned} \tag{1.5}$$

On E we have the vertical distribution $u \to V_uE = \operatorname{Ker} p_{x,u}$, where p_* denotes the differential of p. This consists of vectors which are tangent to fibres and it is locally spanned by $\left(\dot{\partial}_a := \dfrac{\partial}{\partial y^a}\right)$. We shall regard also the vertical distribution as a vector subbundle $VE := \bigcup\limits_{u\in E} V_uE \to E$ of $TE \to E$. Its sections will be called vertical vector fields of E. The tensorial algebra $\mathcal{T}(VE) = \oplus\mathcal{T}^p_q(VE)$, $p, q \in \mathbb{N}$ of this subbundle will be used. Its elements will be indicated by the word "vertical".

Definition 1.1 *A nonlinear connection N on E is a distribution $N : u \to N_uE$, $u \in E$, on E, which is supplementary to the vertical distribution on E.*

We take the distribution N as being locally spanned by $\delta_k = \partial_k - N^a_k(x,y)\dot{\partial}_a$, for $\partial_k := \dfrac{\partial}{\partial x^k}$. By a change of coordinates (1.1), the condition $\delta_k = \dfrac{\partial \widetilde{x}^i}{\partial x^k}\,\widetilde{\delta}_i$ is equivalent with

$$\widetilde{N}^a_j \partial_k \widetilde{x}^j = M^a_b(x)N^b_k(x,y) - \partial_k(M^a_b(x))y^b. \tag{1.6}$$

It is important to notice that from (1.2) it follows that the set of functions $F^a_{bk}(x,y) = \dot\partial_b N^a_k(x,y)$ behaves under a change of coordinates (1.1) as the local coefficients of a linear connection in the vertical bundle over ξ, that is

$$(1.7)\quad \widetilde{F}^a_{bk}(\widetilde{x}(x), \widetilde{y}(x,y)) = M^a_c(x)\widetilde{M}^d_b(\widetilde{x}(x))\frac{\partial x^i}{\partial \widetilde{x}^k}\, F^c_{di}(x,y) - \partial_i(M^a_c(x))\frac{\partial x^i}{\partial \widetilde{x}^k}\, y^c,$$

where $\left(\dfrac{\partial x^i}{\partial \widetilde{x}^k}\right)$ is the inverse matrix of $\left(\dfrac{\partial \widetilde{x}^k}{\partial x^j}\right)$ and $(\widetilde{M}^d_b)$ denotes the inverse matrix of (M^b_c).

We should like to construct a linear connection D in the vertical bundle $VE \to E$. In order to do this it suffices to provide $D_{\delta_k}\dot\partial_a$ and $D_{\dot\partial_a}\dot\partial_b$. Using (1.3) we have the possibility

$$(1.3^\circ)\qquad D_{\delta_k}\dot\partial_a = F^b_{ak}(x,y)\dot\partial_b,\ \ D_{\dot\partial_b}\dot\partial_c = V^a_{bc}(x,y)\dot\partial_a,$$

where necessarily $(V^a_{bc}(x,y))$ behave like the components of a vertical tensor field of type $(1,2)$.

In particular, we may take $V^a_{bc} = 0$ and introduce

Definition 1.2 *The linear connection D in the vertical bundle $VE \to E$ given by*

$$(1.8)\qquad D_{\delta_k}\dot\partial_a = F^b_{ak}(x,y)\dot\partial_b, \quad D_{\dot\partial_a}\dot\partial_b = 0,$$

is called the Berwald connection associated to N.

Definition 1.3 *We call the pair (ξ, N) a Berwald bundle if the functions $F^a_{bk}(x,\, y) = \dot\partial_b N^a_b(x,y)$ depends on x only.*

When (ξ, N) is a Berwald bundle, the functions $F^a_{bk}(x,y) = F^a_{bk}(x)$ define a linear connection ∇ in ξ by

$$(1.9)\qquad \nabla_{\partial_k}\varepsilon_b = F^a_{bk}(x)\varepsilon_a,$$

for (ε_a) a basis of local sections in ξ.

Conversely, if ξ is endowed with a linear connection of local coefficients $F^a_{bk}(x)$, then the functions

$$(1.10)\qquad N^a_k(x,y) = F^a_{bk}(x,y)y^b,$$

define by setting $\delta_k = \partial_k - N^a_k(x,y)\dot\partial_a$ a nonlinear connection on E such that (ξ, N) becomes a Berwald bundle. In other words, any vector bundle endowed with a linear connection is a Berwald bundle.

We notice that the nonlinear connection (1.6) is positively homogeneous of degree 1 in $y = (y^a)$. This suggests us to confine ourselves to the pairs (ξ, N) with the functions $(N^a_k(x,y))$ positively homogeneous of degree 1 in y. The examples to be given later will fall in this category. This assumption requires to eliminate from E the image of the null section as we shall do in the following.

It is well known that, see [8], [9], the Berwald connection induces a covariant derivative in the tensorial algebra of the vertical bundle. This splits in two operators of covariant derivative. The first one is called h–covariant derivative and is defined on functions and vertical vector fields as follows:

$$f_{|k} = \delta_k f,\ X^a_{|k} = \delta_k X^a + F^a_{bk}(x, y) X^b. \tag{1.11}$$

It is extended by usual rules to any vertical tensor field. The second, called the v-covariant derivative, is simply the partial derivative with respect to y

$$f\Big|_a = \dot{\partial}_a f,\ X^a\Big|_b = \dot{\partial}_b X^a, \tag{1.12}$$

since we have chosen $V^a_{bc} = 0$.

We use the notation $_{|k}$ and $\Big|_a$ for denoting the h- and v-covariant derivatives of any vertical tensor field.

Lemma 1.1 *Let ξ be endowed with a positively 1-homogeneous nonlinear connection N and $_{|k}$ the h-covariant derivative defined by the Berwald connection associated to it. Then*

$$y^a_{|k} = 0, \tag{1.13}$$

holds.

Proof. $y^a_{|k} = \delta_k y^a + F^a_{bk}(x, y) y^b = F^a_{bk}(x, y) y^b - N^a_k(x, y) = 0$ because of Euler theorem on homogeneous functions.

Lemma 1.2 *Let (ξ, N) be a Berwald bundle. Then for any vertical tensor field T of local coefficients $T^{a_1 \ldots a_r}_{b_1 \ldots b_s}(x, y)$ we have*

$$T^{a_1 \ldots a_r}_{b_1 \ldots b_s}{}_{|k}\Big|_a = T^{a_1 \ldots a_r}_{b_1 \ldots b_s}\Big|_{a|k}. \tag{1.14}$$

Proof. One verifies (1.10) by a direct calculation keeping in mind that $F^a_{bk} = \dot{\partial}_a N^a_k$ do not depend on y.

2 Berwald bundles endowed with regular Lagrangians

We recall that in $\xi = (E, p, M)$, E means in fact $E \setminus \{(x, 0),\ x \in M\}$.

Definition 2.1 *A smooth function $L : E \to \mathbb{R}$ is called a regular Lagrangian on E if*

(i) *the matrix with the entries $g_{ab}(x, y) = \dfrac{1}{2}\dot{\partial}_a \dot{\partial}_b L$ is nondegenerate,*

(ii) *the quadratic form $g_{ab}(x, y)\zeta^a \zeta^b,\ (\zeta^a) \in \mathbb{R}^m$, is of rank constant.*

A regular Lagrangian L induces a pseudo–Riemannian metric g in the vertical bundle over E, given locally by

$$g(\dot{\partial}_a, \dot{\partial}_b) = g_{ab}(x, y). \tag{2.1}$$

It provides also a set of vertical tensor fields by successively deriving it with respect to (y^a)

$$C_{abc}(x, y) = \frac{1}{4}\dot{\partial}_a\dot{\partial}_b\dot{\partial}_c L,\ D_{abcd}(x, y) = \frac{1}{8}\dot{\partial}_a\dot{\partial}_b\dot{\partial}_c\dot{\partial}_d L,\ \text{etc.} \tag{2.2}$$

Definition 2.2 *Let ξ be endowed with a positively 1-homogeneous nonlinear connection N and with a regular Lagrangian L. We say that N is compatible with L if*

$$L_{|k} := \delta_k L = 0. \tag{2.3}$$

Definition 2.3 *If (ξ, N) is a Berwald bundle with a regular Lagrangian L such that (2.3) holds, the pair (N, L) will be called a Berwald Lagrange structure, shortly a BL structure for ξ.*

Proposition 2.1 *If ξ has a BL structure, then*

(i) $g_{ab|k} = 0,\ C_{abc|k} = 0,\ D_{abcd|k} = 0$ *etc.*

(ii) ${g^{ab}}_{|k} = 0,\ y_{a|k} = 0\ (y_a = g_{ab}y^b),\ C^a_{bc|k} = 0\ (C^a_{ab} = g^{ae}C_{ebc})$.

Proof. Easy consequences of (2.3) and of the commutation formulae (1.10). Assume that ξ has a BL structure.

Let be $c : [0, 1] \to M,\ t \to c(t),\ t \in [0, 1]$ a smooth curve on E. A section A of ξ along c given as $A(t) = A^a(t)\varepsilon_a$ is said to be *parallel* with respect to linear connection ∇ given by $(F^a_{bk}(x))$ if in a local chart on M,

$$\frac{dA^a}{dt} + F^a_{bk}(c(t))A^b(t)\ \frac{dc^k}{dt} = 0, \tag{2.4}$$

holds.

For the initial conditions $c(0) = x$ and $A^a(0) = A^a_0$, the system of differential equations (2.4) admits a unique solution $A^a(x(t))$ and if one assigns to $(A^a_0) \in E_x$ the element $A^a(x(1)) \in E_{c(1)=z}$ one obtains an application $P_c : E_x \to E_z$ called *parallel translation* along c.

Moreover, from the linearity of the system (2.4) it follows that P_c is a linear isomorphism. Now if one considers all loops on M in $x \in M$, the corresponding parallel translations as linear isomorphisms $E_x \to E_x$ provide a group with respect to their composition, called the holonomy group $\phi(x)$ of ∇ in $x \in M$. When M is connected, all these groups are isomorphic and one speaks about the holonomy group ϕ of ∇.

Let L_x be the restriction of L to the fibre E_x. We call L-map a linear isomorphism $f : (E_x, L_x) \to (E_z, L_z)$ with the property $L_x(u) = L_z(f(u))$ for every $u \in E_x$.

Theorem 2.1 *If ξ admits a BL structure, then all parallel translations of ∇ are L-maps. In particular, the holonomy groups $\phi(x)$, $x \in M$, consists of L-maps.*

Proof. Let $c : [0,1] \to M$ be a curve joining the points $x = c(0)$ and $z = c(1)$ of M. Consider a parallel section $A(t) := A(c(t))$, $t \in [0,1]$, of ξ along c. We show that the function $f : t \to L(x(t), A(t))$, $t \in [0,1]$, is constant. Indeed,

$$\frac{dL((x,y),A(t))}{dt} = (\partial_k)\ \frac{dx^k}{dt} + (\dot{\partial}_a L)\ \frac{dA^a}{dt} \overset{(2.4)}{=\!=} L_{|k}\ \frac{dx^k}{dt} = 0.$$

Consider $A_0 \in E_x$ and $A(t)$ the unique solution of (2.4) with the initial condition A_0. Then$P_c(A_0) = A_1$, where $A_1 = A(1)$ and since f is constant, we get $L_x(A_0) = L_z(A_1) = L_z(P_c(A_0))$, q.e.d.

The subset $I_x = \{A \in E_x \mid L_x(A) = 1\}$ of E_x is called the indicatrix of L. Let $G(I_x)$ be the group of all linear isomorphisms of E_x which leave invariant the indicatrix I_x. From Theorem 2.1, it easily follows

Corollary 2.1 *The holonomy group $\phi(x)$ is a subgroup of $G(I_x)$.*

Let us continue to consider a parallel translation $P_c : E_x \to E_z$. Its differential $(P_c)_{*,u}$, $u \in E$ is a linear isomorphism $V_uE \to V_vE$ for $v = P_c(u)$ since P_c itself is a linear isomorphism and $T_u(E_x)$ is nothing but V_uE. We denote it by P_c^v.

In particular, the differentials of the elements of $\phi(x)$ are linear isomorphisms of V_uE with $p(u) = x$ and these provide a group $\phi^v(u)$ that is a subgroup of $GL(V_uE)$.

We call $\phi^v(u)$ the vertical lift of $\phi(x)$. For every $u \in E$, (V_uE, g_u) is a pseudo–Euclidean space.

Theorem 2.2 *The mapping $P_c^v : V_uE \to V_vE$, $v \in P_c(u)$, are linear isometries of pseudo–Euclidean spaces. In particular, the group $\phi^v(u)$ is a subgroups of the isometries of (V_uE, g_u).*

Proof. We fix the curve c joining x, z in M and denote by (P_b^a) the matrix of $P_c : E_x \to E_z$ in the basis $(\varepsilon_a(x))$ and $\varepsilon_a(z)$. Here we tacitly assumed that c is in a domain U of a local chart on M. If it is not so we divide c in segments. The matrix of P_c^v is the same P_b^a in the basis $\dot{\partial}_a\big|_u$ and $\dot{\partial}_a\big|_v$. As P_c is an L-map, we have $L(x, u^a) = L(y, P_b^a u^b)$. We derive this equality two times with respect to (u^a) and we obtain

$$\frac{\partial^2 L}{\partial u^a \partial u^b} = \frac{\partial^2 L}{\partial y^c \partial y^d}\ P_a^c P_b^c,$$

that is $g_{ab}(u) = g_{cd}(v) P_a^c P_b^c$. This exactly means that P_c^v is an isometry of the pseudo–Euclidean spaces (V_uE, g_u) and (V_vE, g_v). q.e.d.

3 Berwald bundles endowed with Finsler functions

Let ξ be a vector bundle.

Definition 3.1 *A smooth function $F : E := E \setminus 0 \to \mathbb{R}$, $(x, y) \to F(x, y)$ is called a Finsler function if*

(i) $F(x, y) \geq 0$,

(ii) $F(x, \lambda y) = \lambda F(x, y), \ \forall \lambda > 0$,

(iii) *the matrix with the entries* $g_{ab}(x, y) = \frac{1}{2} \ \dot{\partial}_a \dot{\partial}_b F^2$ *is positive definite* $(g_{ab}(x,\ y)\zeta^a\zeta^a > 0$ *for* $(\zeta^a) \neq 0)$.

When ξ is endowed with a Finsler function F we call it a vector Finsler bundle. If (ξ, N) is a Berwald bundle, the pair (N, F) will be called a Berwald Finsler structure, shortly a BF structure for ξ if $F_{|k} := \delta_k F = 0$.

If we put $L = F^2$, we obtain a regular Lagrangian. Thus any BF structure is a BL structure. As such, the properties of BL structures proved in the previous section are valid for BF structures. We show new properties for BF structures.

Proposition 3.1 *If ξ has a BF structure then $F_{|k} = 0$ if and only if $g_{ab|k} = 0$.*

Proof. $F_{|k} = 0$ implies $L_{|k} = {F^2}_{|k} = 0$ and by Proposition 2.1, one gets $g_{ab|k} = 0$. Conversely, applying the Euler theorem to F^2 one obtains $F^2(x, y) = g_{ab}(x, y)y^a y^b$. And the h-covariant derivation yields ${F^2}_{|k} = 2FF_{|k} = g_{ab|k}y^a y^b + 2g_{ab}y^a {y^b}_{|k} = 0$ since $y^b_{|k} = 0$. Hence $F_{|k} = 0$. q.e.d.

The pairs (E_x, F_x) are called Minkowski spaces and F_x is called a Minkowski norm on E_x. The reason is that F_x, besides the conditions (i)–(iii) from Definition 3.1 satisfies also (see [5] p.6; (iv) $F_x(y) > 0$ whenever $y \neq 0$; (v) $F_x(y_1 + y_2) \leq F_x(y - 1) + F_x(y_2)$.

The linear isomorphisms of E_x keeping F_x will be called isometries.

We already know by Theorem 2.1 that if ξ has a BF structure, all parallel translations defined by ∇ are isometries.

In particular, the elements of $\phi(x)$ are isometries of the Minkowski space (E_x, F_x). And $\phi(x)$ is a subgroup of the $G(I_x)$, the group of all linear isomorphism which leave invariant the indicatrix I_x.

These facts are basic in the proof of the main result of this section.

Theorem 3.1 *If ξ has a BF structure, the linear connection ∇ is metrizable, that is, there exists a Riemannian metric h in ξ such that $\nabla h = 0$.*

Proof. Let be $x_0 \in M$ and the Minkowski space (E_{x_0}, F_{x_0}). The indicatrix I_x is compact. It follows that the group $G := G(I_x)$ is a compact Lie group. We know that G contains $\phi(x)$ as a Lie subgroup but in general $\phi(x)$ is not

compact. Let $\langle\cdot\rangle$ be an arbitrary inner product in E_{x_0}. Define a new inner product on E_{x_0} by

$$h_{x_0}(u,v) = \frac{1}{\mathrm{Vol}(G)}\int_G < gu, gv > \mu_G, \quad \text{for } g \in G,\ u,v \in E_{x_0},$$

where μ_G denotes the bi–invariant Haar measure on G. It follows that h_{x_0} is G–invariant and, in particular, it is $\phi(x_0)$–invariant, i.e., $h_{x_0}(Pu, Pv) = h_{x_0}(u,v)$ for any $P \in \phi(x_0)$. Now we transfer h_{x_0} to all the points of M. For any point $x \in M$, we consider a curve c joining x with x_0 ($c(0) = x$, $c(1) = x_0$).

Define $h_x(A,B) = h_{x_0}(P_cA, P_cB)$, $A, B \in E_x$. The property that h_{x_0} is $\phi(x_0)$–invariant assures that h_x does not depend on the curve c.

The mapping $h : x \longrightarrow h_x : E_x \times E_x \to R$ is smooth since P_c smoothly depends on x by a general result about dependence of solutions of an ordinary differential equation on initial data. Thus a Riemannian metric h in ξ is obtained. The proof is ended with the help of

Lemma 3.1 *Let h be a Riemannian metric in ξ and $t \to c(t)$, $t \in \mathbb{R}$, a curve with $c(0) = x \in M$. Then*

$$\lim_{t\to 0} \frac{1}{t}\left(h_{c(t)}(P_cA, P_cB) - h_x(A,B)\right) = \left(\nabla_{\dot c(0)} h\right)(A,B)(x), \tag{3.1}$$

where $A, B \in E_x$ and $P_c : E_x \to E_{c(t)}$ is the parallel translation along c.

Indeed, by the definition of h, the term in the left side of (3.1) vanishes. For the proof of Lemma 3.1 we refer to [2].

4 Particular cases

4.1. Let $\xi = \tau_M = (TM, \tau, M)$ be the tangent bundle of M. If τ_M is endowed with a Finsler function F, the pair (M, F) is called a Finsler manifold. For the geometry of these manifolds we refer to [7], [5].

The Finsler function F induces the Cartan nonlinear connection $\overset{\circ}{N}{}^i_j(x,y) = \gamma^i_{j0} - C^i_{jk}\gamma^j_{00}$, where $2\gamma^i_{jk} = g^{ih}(\partial_j g_{kh} + \partial_k g_{jh} - \partial_h g_{jk})$, $2C^i_{jk} = g^{ih}\dot\partial_h g_{jk}$, $\gamma^i_{j0} = \gamma^i_{jk}y^k$ and $\gamma^i_{00} = \gamma^i_{jk}(x,y)y^jy^k$. Of course, $g_{jk} = \dfrac{1}{2}\,\dot\partial_j\dot\partial_k F^2$ denotes the Finsler metric. This nonlinear connection is p–homogeneous of degree 1 in y and is compatible with F, that is, $F_{|k} = 0$. If the local coefficients $\overset{\circ}{G}{}^i_{jk}(x,y) = \dot\partial_j \overset{\circ}{N}{}^i_k(x,y)$ of the Berwald connection associated to $(\overset{\circ}{N}{}^i_j)$ depend on x only, the Finsler manifold (M, F) is called a Berwald space. In [5, p.263–64] there are given five properties characterizing the Berwald spaces. Among them we notice the condition $C_{ijk|h} = 0$. Thus, if τ_M is endowed with a Finsler function F for wich $C_{ijk|h} = 0$, the pair $(\overset{\circ}{N}, F)$ is a Berwald Finsler structure. Particularizing our results from Section 3 the results previously proved by Y. Ichijyo [6] and Z. Szabó [10] are obtained.

4.2. Let $\xi = \tau_M$ be endowed with a regular Lagrangian L. Then the pair (M, L) is called a Lagrange manifold. For the geometry of Lagrange manifolds we refer to [9]. The Lagrangian defines a nonlinear connection $N^i_j(x,t) = \dot{\partial}_j G^i$, where $4G^i = g^{ik}(y^h \dot{\partial}_k \partial_h L - \partial_k L)$ but, in general, this is not p-homogeneous nor compatible with L. We notice that N is provided by the semi–spray $(G^i(x,y))$ that in turn is derived from L. A question is whether there exist Lagrangians which to generate sprays, that is the functions $(G^i(x,y))$ to be p–homogeneous of degree 2 in y.

A first example was given and studied in [3]. A larger class of such Lagrangians called φ–Lagrangians is proposed and studied in [4].

Let τ_M be endowed with a Finsler function F. We eliminate the image of null section $\{0_x,\ x \in M\}$ from TM. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a smooth function. Then $L = \varphi(F^2)$ is a Lagrangian and one proves ([4]) that if $\varphi'(t) \neq 0$ and $\varphi'(t) + t\varphi''(t) \neq 0$ for any $t \in \mathrm{Im}(F^2)$, then L is a regular Lagrangian, called a φ-Lagrangian. For a φ-Lagrangian, the functions $G^i(x,y)$ are p-homogeneous of degree 2 in y. Moreover, $G^i(x,y) = \gamma^i_{00}$ and the nonlinear connection N provided by a φ-Lagrangian coincides with the Cartan nonlinear connection $\overset{\circ}{N}$ of (M, F), cf. [4]. It follows easily that N is compatible with L. Furthermore, the pair (τ_M, N) is a Berwald bundle if and only if $(\tau_M, \overset{\circ}{N})$ is a Berwald bundle. It follows that (N, L) is a Berwald Lagrange structure for τ_M if and only if $(\overset{\circ}{N}, F)$ is a Berwald Finsler structure for τ_M. The connection ∇ is the same for these structures and by Theorem 3.1 it is metrizable.

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MINKOWSKIAN $G-$STRUCTURES IN VECTOR BUNDLES

by Mihai ANASTASIEI

Dedicated to the Memory of Grigorios TSAGAS (1935-2003),
President of Balkan Society of Geometers (1997-2003)

Abstract

A natural generalization of the usual inner product on $\mathbb{R}^m$ is the so-called Minkowski norm. For a smooth vector bundle ξ with the type fibre $\mathbb{R}^m$ endowed with a Minkowski norm, a $G-$structure for ξ, generalizing the $O(m)-$structures, is defined and called a Minkowskian $G-$structure. Several properties of these structures are pointed out in Theorem A and Theorem B. Some of them extend to the vector bundles the results given by Y. Ichijio ([2], [3]) for tangent bundle. If applied to the cotangent bundle our results enrich the geometry of Cartan spaces presented in the monograph [5] by R. Miron et al. on the line of our paper [1].

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Introduction

Let $\xi = (E, \pi, M)$ be a smooth i.e. C^∞vector bundle of rank m. Assume that M is connected. The type fibre of ξ is $\mathbb{R}^m$ and its structural group is $GL(m, \mathbb{R})$. The linear space $\mathbb{R}^m$ has a natural inner product $< , >$ and it is well known that this can be transferred with the help of the bundle charts to a Riemannian structure g in ξ if and only if ξ admits an $O(m)-$structure. Moreover, it is also known that if ξ admits an $O(m)-$structure, then there exists a linear connection ∇ in ξ that is metrical with respect to g $(\nabla g = 0)$. Then ∇ can be also seen as a principal connection in the principal bundle of the frames of ξ having the property that its connection 1-form takes the values in the Lie algebra of $O(m)$.

The condition $\nabla g = 0$ is equivalent with the fact that all parallel translations defined by ∇ are isometries.

Consequently, the fibres $(E_x, g_x), x \in M$, of ξ are all congruent i.e. linearly isometrically isomorphic.

We prove the following theorems.

Theorem A. *If* $\xi = (E, \pi, M)$ *admits a Minkowskian* G_f-*structure, then*

i) *Each fibre E_x, $x \in M$ becomes a Minkowski space,*

ii) *A Finsler function $F(x,y) = f(\mu^a_b(x)y^b)$, $\mu^a_b(x) \in GL(m,\mathbb{R})$ is defined on E.*

Denote by $(g_{ab}(x,y))$ the Finsler metric associated to F.

iii) *Let ∇ be a linear G_f-connection and $|k$ be the horizontal covariant derivative defined by its vertical lift to E. Then $F_{|k} = 0$ and $g_{ab|k} = 0$.*

iv) *the fibres $E_x, x \in M$ are all congruent each others as Minkowski spaces.*

A pair (F, ∇) with F a Finsler function on E and ∇ a linear connection in ξ such that $g_{ab|k} = 0$ will be called a $(F,\nabla)-$**structure** for ξ.

Theorem A says that if ξ admits a G_f- structure, then ξ admits a (F,∇)-structure. The converse holds, too.

Theorem B. *Let $\xi = (E,\pi,M)$ be a vector bundle of rank m. Assume that it admits $(F,\nabla)-$structure. Then F induces a Minkowski norm f on $\mathbb{R}^m$ and ξ admits a G_f-structure such that the $(F,\nabla)-$structure induced by it is just that initially given.*

The notions entering in the contents of these theorems will be explained below in the appropriate places.

Our results extend to any vector bundle some of the results due to Y. Ichijio for tangent bundle, [2], [3].

1 Vector bundles. Minkowskian $G-$structures

Let $\xi = (E,\pi,M)$ be a smooth vector bundle of rank m.

Assume that M is connected and its dimension is n. Then E is a smooth manifold of dimension $n+m$.

Let $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ be a smooth atlas on M. A vector bundle atlas is then $\{(U_\alpha,\varphi_\alpha,\mathbb{R}^m)\}_{\alpha\in A}$, where $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ are diffeomorphisms of the form $\varphi_\alpha(u) = (\pi(u), \varphi_{\alpha,\pi(u)}(u))$, $u \in \pi^{-1}(U_\alpha)$ such that for every $x \in U_\alpha \cap U_\beta \neq \phi$, $\varphi_{\beta,x} \circ \varphi^{-1}_{\alpha,x}$ belongs to $GL(m,\mathbb{R})$.

The manifold structure of E is defined by the atlas $\{(\pi^{-1}(U_\alpha),\phi_\alpha)\}$ with $\phi_\alpha : \pi^{-1}(U_\alpha) \to \psi_\alpha(U_\alpha) \times \mathbb{R}^m$ given by $\phi_\alpha(u) = (\psi_\alpha(\pi(u)), \varphi_{\alpha,\pi(u)}(u))$. The mappings $(\phi_\beta \circ \phi^{-1}_\alpha)(u) = ((\psi_\beta \circ \psi^{-1}_\alpha)(\pi(u)), (\varphi_{\beta,\pi(u)} \circ \varphi^{-1}_{\alpha,\pi(u)})(u))$ are smooth.

Let $(e_a), a = 1,2,...,m$ be the canonical basis of $\mathbb{R}^m$. The mappings $\varepsilon_{\alpha,a} : U_\alpha \to \pi^{-1}(U_\alpha)$, $\varepsilon_{\alpha,a}(x) = \varphi^{-1}_\alpha(x,e_a)$ are m linearly independent local sections of ξ, that is $(\varepsilon_{\alpha,a}(x))$ is a basis of the fibre E_x, $x \in M$.

If we put $\psi_\alpha(x) = (x^i)$, $\psi_\beta(x) = (\widetilde{x}^j)$, $i,j,k... = 1,2,...,n$, then $\psi_\beta \circ \psi^{-1}_\alpha$ has the form

$$(1.1) \qquad \widetilde{x}^j = \widetilde{x}^j(x^1,...,x^n), \operatorname{rank}\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right) = n.$$

For $u \in E$ with $\pi(u) = x$, we can write $u = y^a \varepsilon_{\alpha,a}(x) = \widetilde{y}^b \varepsilon_{\beta,b}(x)$. If we put $(\varphi_{\beta,x} \circ \varphi_{\alpha,x}^{-1})(e_a) = M_a^b(x) e_b$, then $s_{\alpha,a}(x) = M_a^b(x) s_{\alpha,b}$ and $\widetilde{y}^b = M_a^b(x) y^a$.

The local coordinates on E will be (x^i, y^a) and a change of coordinates $(x^i, y^a) \to (\widetilde{x}^i, \widetilde{y}^b)$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \ \mathrm{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n, \tag{1.2}$$
$$\widetilde{y}^a = M_b^a(x) y^b, \ \mathrm{rank}(M_b^a(x)) = m.$$

The Einstein convention on summation will be applied for the indices $i, j, k, ... = 1, ..., n$ as well as for the indices $a, b, c... = 1, ..., m$.

Let G be a Lie subgroup of $GL(m, \mathbb{R})$. One says that ξ admits a **G-structure** if there exists a vector bundle atlas $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}_{\alpha \in A}$ such that the mapping $U_\alpha \cap U_\beta \to GL(m, \mathbb{R})$, $x \to \varphi_{\beta,x} \circ \varphi_{\alpha,x}^{-1}$ take their values in G.

The existence of a $G-$structure in ξ is equivalent with the existence of an open covering $(U_\alpha)_{\alpha \in A}$ of M and of m sections $s_{\alpha,a} : U_\alpha \to \pi^{-1}(U_\alpha)$ for every $\alpha \in A$ such that

i) $(s_{\alpha,a}(x))$ is a basis in E_x, $x \in U_\alpha$,

ii) $s_{\alpha,a}(x) = M_a^b(x) s_{\beta,b}(x)$ with $M_a^b(x) \in G$, for every $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \phi$.

This means that a $G-$structure in ξ is a **reduction to** G of the principal bundle of frames of ξ (see [4]).

The basis $(s_{\alpha,a}(x))$ are called frames **adapted** to the given $G-$structure.

A **Minkowski norm** f on $\mathbb{R}^m$ is a non-negative real function on $\mathbb{R}^m$ with the properties:

1. f is smooth on $\mathbb{R}^m \setminus 0$,
2. $f(\lambda y) = \lambda f(y)$ for all $\lambda > 0$
3. The matrix with the entries $g_{ij}(y) = \dfrac{1}{2}\dfrac{\partial^2 f^2}{\partial y^i \partial y^j}$ is positive definite.

The pair $(\mathbb{R}^m, f)$ is called a **Minkowski space**.

If it happens that $f(-y) = f(y)$, then $f(\lambda y) = |\lambda| f(y)$ and one says that f is an absolutely homogeneous Minkowski norm. One proves [BCS, p.6] that any absolutely homogeneous Minkowski norm is a norm on $\mathbb{R}^m$.

Let $\mathbb{R}^m$ be endowed with a Minkowski norm f and let be $G_f = \{T \in GL(m, \mathbb{R}) | f(Ty) = f(y), \forall y \in \mathbb{R}^m\}$.

Then G_f is a closed subgroup of $GL(m, \mathbb{R})$. Indeed, if $(T_n)_{n \geq 0}$ is a sequence in G_f that converges to T_0, making $n \to \infty$ in the equality $f(T_n y) = f(y)\ \forall y$ we get $f(T_0 y) = f(y)\ \forall y$, that is $T_0 \in G_f$. It follows that G_f is a Lie subgroup of $GL(m, \mathbb{R})$. A G_f-structure in ξ will be called a **Minkowskian structure in** ξ.

2 Finsler vector bundle. Connections

Let $\xi = (E, \pi, M)$ be a smooth vector bundle of rank m.

A **Finsler function** on E is a non-negative real function F on E with the properties:

1. F is smooth on $E \setminus \{(x,0), x \in M\}$ and only continuous on the set $\{(x,0), x \in M\}$,
2. $F(x, \lambda y) = \lambda F(x,y)$ for all $\lambda > 0$,
3. The matrix with the entries $g_{ab}(x,y) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^a \partial y^b}$is positive definite.

The pair (ξ, F) is called a **Finsler vector bundle**.

For every $x \in M$, the function $F_x : E_x \to R$ given by $F_x(u) = F(x,u) \forall u \in E_x$ is a Minkowski norm on E_x. Thus the fibres of a Finsler vector bundle are all Minkowski spaces.

On E we have the vertical distribution $u \to V_u E = \ker \pi_{*,u}$ made by the vectors which are tangent to fibres.

A distribution $u \to H_u E$ which is supplementary to it is called a horizontal distribution or a nonlinear connection for ξ. The vertical distribution is spanned by $\left(\dfrac{\partial}{\partial y^a}\right)$. As a local basis for the horizontal distribution it is usually taken $\delta_i = \dfrac{\partial}{\partial x^i} - N_i^a(x,y)\dfrac{\partial}{\partial y^a}$, where the functions $(N_i^a(x,y))$ are called the local coefficients of a given nonlinear connection. By a change of coordinates on E these functions behave in such a way that the transformation law

$$\text{(2.1)} \qquad \delta_i = \frac{\partial \widetilde{x}^j}{\partial x^i}\widetilde{\delta}_j,$$

is assured.

When a nonlinear connection is considered we have the decomposition

$$\text{(2.2)} \qquad T_u E = H_u E \oplus V_u E,\ u \in E.$$

Then all the geometric objects on E can be decomposed accordingly.

If the functions $(N_i^a(x,y))$ are linear in (y^a), that is $N_i^a(x,y) = \Gamma_{bi}^a(x) y^b$, the nonlinear connection becomes a linear one. In this case we may define an operator of covariant derivative ∇: $\mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$, $(X, \sigma) \to \nabla_X \sigma = X^j(x)\left(\dfrac{\partial \sigma^a}{\partial x^j} + \Gamma_{bj}^a(x)\sigma^b\right)\varepsilon_a$, where $X = X^j \dfrac{\partial}{\partial x^j}$, $\sigma = \sigma^a(x)\varepsilon_a$ and $\nabla_{\frac{\partial}{\partial x^k}} \varepsilon_a = \Gamma_{ak}^b(x)\varepsilon_b$.

We denoted by $\Gamma(E)$ the $\mathcal{F}(M)-$module of sections in ξ and $\mathcal{X}(M) := \Gamma(TM)$.

A linear connection ∇ in ξ induces a linear connection D in the vertical bundle over E as follows:

The operator D: $\mathcal{X}(E) \times \Gamma(VE) \to \Gamma(VE)$ is defined by the equations

(2.3) $\qquad D_{\delta_k}\dot{\partial}_a = \Gamma^b_{ak}(x)\dot{\partial}_b,\ D_{\dot{\partial}_b}\dot{\partial}_a = V^c_{ab}(x,y)\dot{\partial}_c,$

where $\dot{\partial}_a := \dfrac{\partial}{\partial y^a}$ and $C^c_{ab}(x,y)$ are the components of a vertical tensor field $V^c_{ab}\dot{\partial}_c \otimes \delta y^a \otimes \delta y^b$, $\delta y^a = dy^a + \Gamma^a_{b_i}(x)y^b dx^i$.

The functions C^c_{ab} can be taken zero. In such a case, D will be called the vertical lift of ∇.

We shall use D_{δ_k} for defining a **horizontal covariant derivative** in the tensor algebra of the vertical bundle over E. It will be denoted by $|k$ and it is obtained by the usual extension procedure starting with

(2.4) $\qquad f_{|k} = \delta_k f$ for every real function f on E,

$A^a_{|k} = \delta_k A^a + \Gamma^a_{bk}(x)A^b$, for $A = A^a\dot{\partial}_a$ a section in the vertical bundle.

The vector field $C = y^a\dot{\partial}_a$ is called the Liouville vector field on E and $D_{\delta_k}C = y^a_{|k}\dot{\partial}_a$ is called the deflexion tensor field of D. We have

Lemma 2.1 $D_{\delta_k}C = 0$.

Indeed, $y^a_{|k} = \delta_k y^a + \Gamma^a_{b_k}(x)y^b = -\Gamma^a_{b_k}(x)y^b + \Gamma^a_{b_k}(x)y^b = 0$.

Lemma 2.2 *For every real function H on E we have*

$$\dot{\partial}_a(H_{|k}) = (\dot{\partial}_a H)_{|k}.$$

Proof. A direct calculation keeping in mind that $\dot{\partial}_a H$ are he coefficients of a vertical 2-form and so

$$(\dot{\partial}_a H)_{|k} = \delta_k(\dot{\partial}_a H) - \Gamma^b_{ak}(x)\dot{\partial}_b H.$$

3 Proof of Theorem A

Let be ξ with the type fibre the Minkowski space $(\mathbb{R}^m, f)$. Assume that ξ admits a G_f-structure. Let $(s_{\alpha,a}(x))$ be a frame in E_x adapted to this G_f-structure. For $u \in E_x$ we have $u = y^a\varepsilon_{\alpha,a}(x) = z^a s_{\alpha,a}(x)$. We define $F_\alpha : E_x \to [0,\infty)$ by $F_\alpha(u) = f(z^a)$. For $x \in U_\alpha \cap U_\beta$ we have also $F_\beta(u) = f(\widetilde{z}^b)$, where $(\widetilde{z}^b)$ are given by $u = \widetilde{z}^b s_{\beta,b}(x)$. It follows that $\widetilde{z}^b = M^b_a(x)z^a$ with $(M^b_a(x)) \in G_f$. Consequently, $f(\widetilde{z}^b) = f(z^a)$ and $F_\alpha(u) = F_\beta(u)$. In the other words, the function F defined by $F(u) = f(z^a)$ does not depend on the chosen local chart. It is clear that for every $x \in M$, this F is a Minkowski norm on E_x. Thus i) of Theorem A is proved.

If we put $s_{\alpha,a}(x) = \lambda^b_a(x)\varepsilon_{\alpha,b}(x), (\lambda^b_a(x)) \in GL(m,\mathbb{R})$, it results $z^a = \mu^a_b(x)y^b$ with $(\mu^a_b) = (\lambda^a_b)^{-1}$.

The function $F : (x,y) \to F(x,y) = f(\mu^a_b(x)y^b)$ is a Finsler function on E. Indeed, this is smooth on $E \setminus \{(x,0), x \in M\}$ since f is smooth on $\mathbb{R}^m \setminus 0$ and the functions (μ^a_b) as the entries of the inverse matrix of a matrix whose entries are smooth are also smooth.

Note that $y^a = 0$ if and only if $z^a = 0$. The function F is positively homogeneous of degree 1 in (y^a) because f is homogeneous of degree 1. The matrix $(g_{ab}(x,y))$ has in this case the entries $g_{ab}(x,y) = \dfrac{1}{2}\dfrac{\partial^2 f^2}{\partial y^c \partial y^d}\mu_a^c(x)\mu_b^d(x)$ and for any $(\varsigma^a) \in \mathbb{R}^m$ we get

$$g_{ab}(x,y)\varsigma^a\varsigma^b = \frac{1}{2}\frac{\partial^2 f^2}{\partial y^c \partial y^d}\sigma^c\varsigma^d \text{ for } \varsigma^c = \mu_a^c(x)\xi^a.$$

It follows that the matrix (g_{ab}) is positive definite since f is a Minkowski norm. Thus ii) of Theorem A is proved. Note that ii) implies i) because the Finsler function F will provide by restriction a Minkowski norm in each fibre of ξ.

We fix an open set U_α in M and take a field of frames (s_a) of local sections adapted to the Minkowskian structure G_f. A G_f-connection is a connection in the principal bundle of frames of ξ whose connection 1-form θ has values in the Lie algebra g_f of G_f. The 1-form θ is completely determined by a matrix $(\theta_b^a(x))$ of 1-forms on M using a fixed basis of g_f. The operator of covariant derivative is $\nabla_X^* s_a = \theta_a^b(X)s_b$. If one sets $\theta_a^b = \Gamma_{ak}^{*b}(x)dx^k$, then $\theta_a^b(X) = X^k\Gamma_{ak}^{*b}(x)$ for $X = X^k\partial_k$ and so $\nabla_X^* s_a = \Gamma_{ak}^{*b}(x)X^k s_b$ with the matrix $(\Gamma_{ak}^{*b}(x)X^k)$ in g_f. The following Lemma gives a characterization of the elements of g_f.

Lemma 3.1 *A matrix $A = (A_b^a) \in g_f$ if and only if*

$$\frac{\partial f}{\partial z^a}A_b^a z^b = 0 \textit{ for every } (z^a) \in \mathbb{R}^m. \tag{3.1}$$

Proof. If $A \in g_f$ then $\exp tA \in G_f$, hence $f((\exp tA)z) = f(z)\ \forall z \in \mathbb{R}^m$. This means that $\dfrac{d}{dt}f((\exp tA)z)|_{t=0} = 0$, a equation that is equivalent with (3.1).

Note that for $f = \sqrt{\langle z, z\rangle}$, the equation (3.1) reduces to the skew symmetry of A.

Let (ε_a) be the natural frame (corresponding to the canonical basis (e_a) in $\mathbb{R}^m$) on U_α so that $u = y^a\varepsilon_a = z^b s_b$. It result $z^b = \mu_c^b y^c$ for $\mu_c^b = (\lambda_b^a)^{-1}$, where as before $s_a(x) = \lambda_a^b(x)\varepsilon_b(x)$. We put $\nabla^*\varepsilon_a = X^k\Gamma_{ak}^b(x)\varepsilon_b$.

Then $\nabla_X^* s_a = \nabla_X^*(\lambda_a^b\varepsilon_b) = X^k(\partial_k\lambda_a^c + \Gamma_{bk}^c(x)\lambda_a^b)\varepsilon_c$.

If we think (λ_a^b) as a set of m vector fields, the last parenthesis is $\nabla_k\lambda_a^c$, hence $\nabla_X^k\varepsilon_a = X^k(\nabla_k\lambda_a^c)\varepsilon_c$.

On the other hand, $\nabla_X^* s_a = X^k\Gamma_{ak}^{*b}\lambda_b^c\varepsilon_c$ and by a comparison we get $\nabla_k\lambda_a^c = \Gamma_{ak}^{*b}\lambda_b^c$ or

$$\Gamma_{ak}^{*b} = (\nabla_k\lambda_a^c)\mu_c^b. \tag{3.2}$$

Lemma 3.1 applied for $(X^k\Gamma_{ak}^{*b})$ says that

$$\frac{\partial f}{\partial z^b}(X^k\Gamma_{ak}^{*b})z^a = 0\ \forall (z^a) \in \mathbb{R}^m. \tag{3.3}$$

Inserting here Γ^{*b}_{ak} given by (3.2) one gets

$$\frac{\partial f}{\partial z^b}(\nabla_k \lambda^c_a)\mu^b_c z^a = 0. \tag{3.4}$$

Now we consider the nonlinear connection $N^a_k(x,y) = \Gamma^a_{bk}(x)y^b$ and the vertical lift of linear connection $(\Gamma^a_{bk}(x))$ denoting by $|i$ the corresponding horizontal covariant derivative.

Recall that $F(x,y) = f(\mu^a_b(x)y^b)$ and compute $F_{|k}$.

We have

$$F_{|k} = \frac{\partial F}{\partial x^k} - N^a_k \frac{\partial F}{\partial y^a} = \frac{\partial f}{\partial z^c}\frac{\partial \mu^c_b}{\partial x^k}y^b - \Gamma^a_{bk}y^b \frac{\partial f}{\partial z^c}\mu^c_a = \frac{\partial f}{\partial z^c}y^b\left(\frac{\partial \mu^c_b}{\partial x^k} - \Gamma^a_{bk}\mu^c_a\right).$$

From $\mu^c_b\lambda^a_c = \delta^a_b$ it follows

$$\frac{\partial \mu^c_b}{\partial x^k}\lambda^a_c = -\mu^c_b\frac{\partial \lambda^a_c}{\partial x^k} \text{ or } \frac{\partial \mu^c_b}{\partial x^k} = -\mu^d_b\frac{\partial \lambda^a_d}{\partial x^k}\mu^c_a.$$

Inserting this in the last form of $F_{|k}$ we get

$$F_{|k} = -\frac{\partial f}{\partial z^c}y^b\mu^d_b\left(\frac{\partial \lambda^a_d}{\partial x^k} + \Gamma^a_{ek}\lambda^e_d\right)\mu^c_a = -\frac{\partial f}{\partial z^c}z^d(\nabla_k\lambda^a_d)\mu^c_a = 0$$

by (3.4).

Using Lemma 2.2, we compute

$$g_{ab|k} = \frac{1}{2}(\dot\partial_a\dot\partial_b F^2)_{|k} = \frac{1}{2}\dot\partial_a\dot\partial_b(F^2_{|k}) = 0$$

since $F_{|k} = 0 \Leftrightarrow F^2_{|k} = 0$.

Thus the point iii) of Theorem A is proved.

Let us consider a smooth curve $c : [0,1] \to M; t \to c(t)$, joining the points $x = c(0)$ and $y = c(1)$ and let us denote by $P_c : E_x \to E_y$ the parallel translation along c defined by a linear G_f-connection ∇ in ξ.

It associates to an element $u = A(0) \in E_x$ the unique element $A(1)$ from E_y, where $t \to A(t)$ is a section in ξ along c which is parallel along c, that is its components $(A^a(t))$ are solutions of the system of differential equations

$$\frac{dA^a}{dt} + \Gamma^a_{bk}(x(t))A^b(t)\frac{dx^k}{dt} = 0. \tag{3.5}$$

Consider F in the points $(x(t), A(t))$ and compute

$$\frac{dF(x(t),A(t))}{dt} = \frac{\partial F}{\partial x^k}\frac{dx^k}{dt} + \frac{\partial F}{\partial y^a}\frac{dA^a}{dt} \overset{(3.5)}{=} \frac{dx^k}{dt}\left(\frac{\partial F}{\partial x^k} - \Gamma^a_{bk}A^b\frac{\partial F}{\partial y^a}\right) = 0$$

because of $F_{|k} = 0$.

Thus the function $t \to F(x(t), A(t))$ is constant. Hence $F_x(u) = F_y(P_c u)$. In the other words, the linear isomorphism P_c preserves the Minkowskian norms. This proves the point (iv) and thus Theorem A is completely proved.

4 Proof of Theorem B

Let be a pair (F,∇) with F a Finsler function on E and ∇ a linear connection in ξ such that $g_{ab|k}=0$.

Lemma 4.1 $g_{ab|k}=0$ *implies* $F_{|k}=0$.

Proof. The homogeneity of F implies by a repeated use of the Euler theorem that $F^2(x,y)=g_{ab}(x,y)y^ay^b$.

Then $F^2_{|k} = g_{ab|k}y^ay^b + 2g_{ab}y^a_{|k}y^b = 0$, by hypothesis and Lemma 2.1. Hence $F_{|k}=0$, q.e.d.

We have proved in the end of Section 3 that if $F_{|k}=0$, then all parallel translations of ∇ are isometries of Minkowski spaces.

In particular, the holonomy group, let say H, of ∇ is made of isometries of Minkowski spaces.

Let $\mathcal{H}$ be the Lie algebra of H and an element $A=(A^a_b)\in\mathcal{H}$. Then $\exp tA\in H$ and we have

$$F(x,(\exp tA)y)=F(x,y),\forall x\in M,\forall y\in E_x. \tag{4.1}$$

This is equivalent with

$$\frac{d}{dt}F(x,(\exp tA)y)|_{t=0}=0. \tag{4.1'}$$

The linear connection ∇ in ξ corresponds to an infinitesimal connection in the principal bundle of linear frames of ξ.

By the Holonomy Theorem ([4]) this principal bundle admits an $H-$ structure (a reduction to the Lie subgroup H) such that Γ becomes an $H-$ connection.

Correspondingly, ξ admits a reduction to H such that ∇ is an $H-$ connection.

Let be $s_a=\lambda^b_a(x)\varepsilon_a$ a field of frames on the open set U_α containing x. We think (4.1) and (4,1') in this frame taking $y=y^a\varepsilon_a=\varepsilon^a s_a$. Thus $\dfrac{d}{dt}((\exp tA)y))|_{t=0}=A_y=(A^a_b\xi^b)s_a=A^a_b\mu^b_c y^c\lambda^d_a\varepsilon_d.$

Expanding (4.1') we find

$$(\dot\partial_d F)\lambda^d_a A^a_b\mu^b_c y^c=0, \tag{4.2}$$

where $\dot\partial_d F$, $d=1,...,m$ mean the partial derivatives with the second set of m variables of $F(\cdot,\cdot)$.

When we put $\nabla_X s_a = X^k\Gamma^{*b}_{ak}s_b$, we necessarily have $(X^k\Gamma^{*b}_{ak})\in\mathcal{H}$.

In the natural frame we set $\nabla_X\varepsilon_a=X^k\Gamma^b_{ak}\varepsilon_b(x)$ and as before we get

$$\nabla_X s_a = X^k(\nabla_k\lambda^c_a)\mu^b_c s_b(x). \tag{4.3}$$

Thus by comparison it follows (3.2).

Now we write (4.2) for the matrix $(X^k\Gamma^{*a}_{bk})$. We get

$$(\dot\partial_d F)(\nabla_k\lambda^d_a)\mu^a_c y^c=0. \tag{4.4}$$

Let be $F(x^i, y^a) = F(x^i, \lambda^a_b(x)\xi^b) := f(x, \xi)$.
We show that f does not depend on x.
We compute

$$\partial_k F = \partial_k F + (\dot{\partial}_a F)\partial_k(\lambda^a_b(x))\xi^b \overset{F_{|k}=0}{=} y^b \Gamma^a_{bk}\dot{\partial}_a F + (\dot{\partial}_a F)\partial_k(\lambda^a_b(x))\xi^b =$$

$$= (\dot{\partial}_a F)(\partial_k \lambda^a_c + \Gamma^a_{bk}\lambda^b_c)\xi^c = (\dot{\partial}_a F)(\nabla_k \lambda^a_c)\mu^c_e y^e = 0,$$

by (4.4).

Thus $F(x^i, y^i) = F(x, \lambda^a_b(x)\xi^b) = f(\xi^a)$.

We regard f as a function on $\mathbb{R}^m$ and it obvious that f is a Minkowski norm.

Now we show that the holonomy group $H \subset G_f$. Let $T \in H$ with $(T^{*a}_{\ \ b})$ its matrix in the frame (s_a) and (T^a_b) its matrix in the frame (ε_a). Then $T^{*a}_{\ \ b} = \mu^a_c T^c_d \lambda^d_b$. We have $f(\xi^a) = f(\mu^a_b y^b) = F(x^i, y^a) = F(x^i, T^a_b y^b) = f(\mu^c_a T^a_b y^b) = f(\mu^c_a \lambda^a_e T^{*e}_{\ \ d} \mu^d_b y^b) = f(T^{*c}_{\ \ d}\xi^d)$. Thus $T \in G_f$.

As ξ admits an $H-$structure, we may say that it admits also a G_f-structure. If one reviews the proof of Theorem A it comes out that the $(\nabla, F)-$ structure induced by this G_f-structure is just that initially given.

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SOME NEW PROPERTIES OF BERWALD - CARTAN SPACES

by Mihai ANASTASIEI[1]

Abstract

A manifold endowed with a regular Hamiltonian which is 2–homogeneous in momenta was called a *Cartan space*.The geometry of regular Hamiltonians as smooth functions on the cotangent bundle is mainly due to R. Miron and it is now systematically described in the monograph [4]. An interesting particular class of Cartan spaces is given by the so–called Berwald–Cartan spaces. In this paper some new properties of the Berwald–Cartan spaces are proved.
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Introduction

Analytical Mechanics and some theories in Physics brought into discussion regular Lagrangians and their geometry, [5]. A regular Lagrangian which is 2-homogeneous in velocities is nothing but the square of a fundamental Finsler function and its geometry is Finsler geometry. This geometry was developed since 1918 by P. Finsler, E. Cartan, L. Berwald and many others, see [2] and the most recent graduate text [1]. But in Mechanics and Physics there exists also regular Hamiltonians whose geometry is also useful. This geometry is mainly due to R. Miron, [3], and it is now systematically presented in the monograph [4]. A manifold endowed with a regular Hamiltonian which is 2-homogeneous in momenta was called a Cartan space. The notion of Cartan space was introduced by R. Miron in [3]. A particular and interesting class of Cartan spaces is given by the so–called Berwald–Cartan spaces, shortly BC-spaces. The geometry of the BC-spaces can be found in [4], Chs. 6-7. Our purpose is to prove some new properties of these spaces. A Cartan space is a pair (M, K) for M a smooth manifold and K a regular Hamiltonian which is 2-homogeneous in momenta. A BC space is defined as a Cartan space whose Chern–Rund connection coefficients of the canonical metrical connection do not depend on momenta, that is, $H^i_{jk}(x,p) = H^i_{jk}(x)$. For a Cartan space the pair $(T^*_xM, K(x,p))$ for any fixed $x \in M$ is a Minkowski space. We prove (Theorem 3.2) that for BC spaces the Minkowski spaces $(T^*_xM, K(x,p))$ are

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all linearly isometric to each other. Noticing that the functions $H^i_{jk}(x)$ defines a symmetric linear connection ∇ on M we prove (Theorem 3.3) that ∇ is metrizable, that is, there exists a Riemannian metric on M whose Levi–Civita connection is ∇. These proofs are presented in Section 3. Some preliminaries from the geometry of cotangent bundle are given in Section 1, and Section 2 contains necessary facts from the geometry of Cartan spaces.

1 Preliminaries

Let M be an n-dimensional C^∞ manifold and $\tau^* : T^*M \to M$ its cotangent bundle. If (x^i) are local coordinates on M, then (x^i, p_i) will be taken as local coordinates on T^*M with the momenta (p_i) provided by $p = p_i dx^i$ where $p \in T^*_x M$, $x = (x^i)$ and (dx^i) is the natural basis of $T^*_x M$. The indices $i, j, k...$ will run from 1 to n and the Einstein convention on summation will be used. A change of coordinates $(x^i, p_i) \to (\widetilde{x}^i, \widetilde{p}_i)$ on T^*M has the form

$$(1.1)\qquad \begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \quad \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}(\widetilde{x}) p_j, \end{aligned}$$

where $\left(\dfrac{\partial x^j}{\partial \widetilde{x}^i}\right)$ is the inverse of the Jacobian matrix $\left(\dfrac{\partial \widetilde{x}^j}{\partial x^k}\right)$.

Let $\left(\partial_i := \dfrac{\partial}{\partial x^i},\ \partial^i := \dfrac{\partial}{\partial p_i}\right)$ be the natural basis in $T_{(x,p)}T^*M$. The change of coordinates (1.1) produces

$$(1.2)\qquad \begin{aligned} &\partial_i = (\partial_i \widetilde{x}^j)\widetilde{\partial}_j + (\partial_i \widetilde{p}_j)\widetilde{\partial}^j, \\ &\widetilde{\partial}^i = (\partial_j \widetilde{x}^i)\partial^j. \end{aligned}$$

The natural cobasis (dx^i, dp_i) from $T^*_{(x,p)}T^*M$ transforms as follows.

$$(1.3)\qquad d\widetilde{x}^i = (\partial_j \widetilde{x}^i) dx^j,\ d\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\, dp_j + \frac{\partial^2 x^j}{\partial \widetilde{x}^i \partial \widetilde{x}^k}\, p_j\, dx^k.$$

The kernel $V_{(x,p)}$ of the differential $d\tau^* : T_{(x,p)}T^*M \to T_x M$ is called the *vertical* subspace of $T_{(x,p)}T^*M$ and the mapping $(x,p) \to V_{(x,p)}$ is a regular distribution on T^*M called the *vertical distribution.* This is integrable with the leaves $T^*_x M$, $x \in M$ and is locally spanned by (∂^i). The vector field $C^* = p_i \partial^i$ is called the Liouville vector field and $\omega = p_i dx^i$ is called the Liouville 1-form on T^*M. Then $d\omega$ is the canonical symplectic structure on T^*M. For an easier handling of the geometrical objects on T^*M it is usual to consider a supplementary distribution to the vertical distribution, $(x,p) \to N_{(x,p)}$, called the *horizontal distribution* and to report all geometrical objects on T^*M to the decomposition

$$(1.4)\qquad T_{(x,p)}T^*M = N_{(x,p)} \oplus V_{(x,p)}.$$

The pieces produced by the decomposition (1.4) are called d–geometrical objects (d is for distinguished) since their local components behave like geometrical objects on M, although they depend on $x = (x^i)$ and momenta $p = (p_i)$.

The horizontal distribution is taken as being locally spanned by the local vector fields

$$\delta_i := \partial_i + N_{ij}(x,p)\partial^j, \tag{1.5}$$

and for a change of coordinates (1.1), the condition

$$\delta_i = (\partial_i \widetilde{x}^j)\widetilde{\delta}_j \text{ for } \widetilde{\delta}_j := \widetilde{\partial}_j + \widetilde{N}_{jk}(\widetilde{x},\widetilde{p})\widetilde{\partial}^k, \tag{1.6}$$

is equivalent with

$$\widetilde{N}_{ij}(\widetilde{x},\widetilde{p}) = \frac{\partial x^s}{\partial \widetilde{x}^i}\,\frac{\partial x^r}{\partial \widetilde{x}^j}\,N_{sr}(x,p) + \frac{\partial^2 x^r}{\partial \widetilde{x}^i \partial \widetilde{x}^r}\,p_r. \tag{1.7}$$

The horizontal distribution is called also a *nonlinear connection* on T^*M and the functions (N_{ij}) are called the local coefficients of this nonlinear connection. It is important to note that any regular hamiltonian on T^*M determines a nonlinear connection whose local coefficients verify $N_{ij} = N_{ji}$. The basis (δ_i, ∂^i) is adapted to the decomposition (1.4). The dual of it is $(dx^i, \delta p_i)$, for $\delta p_i = dp_i - N_{ji}dx^j$ and then $\delta \widetilde{p}_i = \dfrac{\partial x^j}{\partial \widetilde{x}^i}\,\delta p_j$.

2 Cartan spaces

A *Cartan structure* on M is a function $K : T^*M \to [0,\infty)$ with the following properties:

1. K is C^∞ on $T^*M \setminus 0$ for $0 = \{(x,0),\ x \in M\}$,
2. $K(x,\lambda p) = \lambda K(x,p)$ for all $\lambda > 0$,
3. The $n \times n$ matrix (g^{ij}), where $g^{ij}(x,p) = \dfrac{1}{2}\,\partial^i\partial^j K^2(x,p)$, is positive–definite at all points of $T^*M \setminus 0$.

We notice that in fact $K(x,p) > 0$, whenever $p \neq 0$.

Definition 2.1. The pair (M,K) is called a *Cartan space.*

Example. Let $(\gamma_{ij}(x))$ be the matrix of the local coefficients of a Riemannian metric on M and $(\gamma^{ij}(x))$ its inverse. Then $K(x,p) = \sqrt{\gamma^{ij}(x)p_i p_j}$ gives a Cartan structure. Thus any Riemannian manifold can be regarded as a Cartan space. More examples can be found in Ch. 6 of [4].

We put $p^i = \dfrac{1}{2}\,\partial^i K^2$ and $C^{ijk} = -\dfrac{1}{4}\,\partial^i\partial^j\partial^k K^2$. The properties of K imply

$$\begin{aligned} &p^i = g^{ij}p_j,\ p_i = g_{ij}p^j,\ K^2 = g^{ij}p_ip_j = p_ip^j,\\ &C^{ijk}p_k = C^{ikj}p_k = C^{kij}p_k = 0. \end{aligned} \tag{2.1}$$

One considers the *formal Christoffel symbols*

$$\gamma^i_{jk}(x,p) := \frac{1}{2}\, g^{is}(\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{jk}) \tag{2.2}$$

and the contractions $\gamma^\circ_{jk}(x,p) := \gamma^i_{jk}(x,p)p_i$, $\gamma^\circ_{j\circ} := \gamma^i_{jk}p_i p^k$. Then the functions

$$N_{ij}(x,p) = \gamma^\circ_{ij}(x,p) - \frac{1}{2}\, \gamma^\circ_{h\circ}(x,p)\partial^h g_{ij}(x,p), \tag{2.3}$$

verify (1.7). In other words, these functions define a nonlinear connection on T^*M. This nonlinear connection was discovered by R. Miron, [3]. Thus a decomposition (1.4) holds. From now on we shall use only the nonlinear connection given by (2.3).

A linear connection D on T^*M is said to be an N*–linear connection* if

1° D preserves by parallelism the distributions N and V,

2° $D\theta = 0$, for $\theta = \delta p_i \wedge dx^i$.

One proves that an N-linear connection can be represented in the adapted basis (δ_i, ∂^i) in the form

$$\begin{aligned} D_{\delta_j}\delta_i &= H^k_{ij}\delta_j, \quad D_{\delta_j}\partial^i = -H^i_{kj}\partial^k,\\ D_{\partial^j}\delta_i &= V_i^{kj}\delta_k, \quad D_{\partial^j}\partial^i = -V_k^{ij}\delta^k, \end{aligned} \tag{2.4}$$

where V_i^{kj} is a d–tensor field and $H^k_{ij}(x,p)$ behave like the coefficients of a linear connection on M. The functions H^k_{ij} and V_i^{kj} define operators of h–covariant and v-covariant derivatives in the algebra of d-tensor fields, denoted by ${}_{|k}$ and $\big|^k$, respectively. For g^{ij} these are given by

$$\begin{aligned} g^{ij}{}_{|k} &= \delta_k g^{ij} + g^{sj}H^i_{sk} + g^{is}H^j_{sk},\\ g^{ij}\big|^k &= \partial^k g^{ij} + g^{sj}V_s^{ik} + g^{is}V_s^{jk}. \end{aligned} \tag{2.5}$$

An N-linear connection given in the adapted basis (δ_i, ∂^j) as $D\Gamma(N) = (H^i_{jk}, V_j^{ik})$ is called *metrical* if

$$g^{ij}{}_{|k} = 0, \quad g^{ij}\big|^k = 0. \tag{2.6}$$

One verifies that the N-linear connection $C\Gamma(N) = (H^i_{jk}, C_i^{jk})$ with

$$\begin{aligned} H^i_{jk} &= \frac{1}{2}\, g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}),\\ C_i^{jk} &= -\frac{1}{2}\, g_{is}(\partial^j g^{sk} + \partial^k g^{sj} - \partial^s g^{jk}) = g_{is}C^{sjk}, \end{aligned} \tag{2.7}$$

is metrical and its h-torsion $T^i_{jk} := H^i_{jk} - H^i_{kj} = 0$, v-torsion $S^{jk}_i := C^{jk}_i - C^{kj}_i = 0$ and the deflection tensor $\Delta_{ij} = N_{ij} - p_k H^k_{ij} = 0$. Moreover, it is unique with these properties. This is called the canonical metrical connection of the Cartan space (M, K). It has also the following properties:

$$
\begin{aligned}
&K_{|j} = 0, \quad K\big|^j = \frac{p^j}{K}, \quad K^2{}_{|j} = 0, \quad K^2\big|^j = 2p^j, \\
&p_{i|j} = 0, \quad p_i\big|^j = \delta^j_i, \quad p^i{}_{|i} = 0, \quad p^i\big|^j = g^{ij}.
\end{aligned}
\tag{2.8}
$$

Besides $C\Gamma(N)$ one may consider on T^*M three other important N-linear connection which are partially or not at all metrical: Chern–Rund connection $CR\Gamma(N) = (H^i_{jk}, 0)$, the Hashiguchi connection $H\Gamma(N) = (\partial^i N_{jk}, C^{kj}_i)$ and the Berwald connection $B\Gamma(N) = (\partial^i N_{jk}, 0)$.

3 Berwald–Cartan spaces

Let $C^n = (M, K)$ be a Cartan space with the canonical metrical connection $C\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ given by (2.7).

Definition 3.1. The Cartan space C^n is called a *Berwald–Cartan space*, shortly a BC space, if the connection coefficients H^i_{jk} do not depend on momenta, that is, $H^i_{jk}(x, p) = H^i_{jk}(x)$.

In [4], by direct methods or using the duality between Finsler and Cartan spaces given by the Legendre map, one proves

Theorem 3.1. *The following assertions are equivalent:*

1° *The Cartan space C^n is a BC space,*

2° *The coefficients $B^i_{jk} = \partial^i N_{jk}$ of the Berwald connection are functions of position only, that is $B^i_{jk}(x, p) = B^i_{jk}(x)$,*

3° *The curvature $P_j{}^i{}_k{}^h := \dot\partial^h B^i_{jk}$ of the Berwald connection vanishes.*

4° $C^{ijk}{}_{|h} = 0$.

For the Cartan space $C^n = (M, K)$, the function $K_x := K(x, \cdot) : T^*_x M \to \mathbb{R}$ is a *Minkowski norm* for every $x \in M$. Thus we have the Minkowski spaces $(T^*_x M, K_x)$, $x \in M$. For BC spaces, the following theorem holds.

Theorem 3.2. *Let (M, K) be a BC space. Whenever M is connected the Minkowski spaces $(T^*_x M, K_x)$ are all linearly isometric to each other.*

Proof. Let $\omega = \omega_i dx^i$ an 1-form and $v = v^j \partial_j$ a vector field on M. Using the connection coefficients $H^i_{jk}(x)$ we may define a covariant derivative of ω in the direction of v as follows: $\nabla_v \omega = v^k(\partial_k \omega_i - H^j_{ik}\omega_j)dx^i$.

We restrict ω to a curve $c : t \to x(t)$, $t \in \mathbb{R}$, on M, define the covariant derivative of ω along c by $\dfrac{\nabla\omega}{dt} = \left[\dfrac{d\omega_i}{dt} - H^j_{ik}\omega_j \dfrac{dx^k}{dt}\right] dx^i$ and we say that ω is parallel along c if $\dfrac{\nabla\omega}{dt} = 0$. Let us estimate $\dfrac{dK^2(x(t), \omega(t))}{dt}$. We write the equality $K^2(x,p) = g^{ij}(x,p)p_j p_j$ for $(x(t), \omega(t))$ and we obtain that along the curve c: $\dfrac{dK^2}{dt} = \dfrac{dg^{ij}}{dt}\omega_i\omega_j + 2g^{ij}\omega_i \dfrac{d\omega_j}{dt}$. But $\dfrac{d}{dt}(g^{ij}) = (\delta_k g^{ij})\dfrac{dx^k}{dt} + (\partial^k g^{ij})\dfrac{\delta p_k}{dt}$ and using $g^{ij}{}_{|k} = 0$ as well as the last equation (2.1) one gets:

$$\frac{dK^2}{dt} = 2g^{ij}\omega_i \left(\frac{d\omega_j}{dt} - H^s_{jk}\omega_s \frac{dx^k}{dt}\right).$$

From here we read

Lemma 3.1. *If the* 1*-form* ω *is parallel along the curve* $c : t \to x(t)$*, then the function* $K(t) := K(x(t), \omega(t))$ *is constant along the curve* c*.*

Let x, y be points of M joined by a curve $c : [0,1] \to M$ such that $c(0) = x$, $c(1) = y$. Let be $\alpha \in T^*_x M$. We consider the unique solution $\omega = (\omega_i)$ of the system of linear ordinary differential equations $\dfrac{d\omega_i}{dt} - H^j_{ik}\omega_j \dfrac{dx^k}{dt} = 0$ with the initial condition $\omega(0) = \alpha$ and we associate to α the element $\alpha' = \omega(1)$ of $T^*_y M$. The mapping $T^*_x M \to T^*_y M$ given by $\alpha \to \alpha'$ is a linear isomorphism. By Lemma 3.1, $K(x(t), \omega(t))$ has the same values at $t = 0$. Hence $K_x(\alpha) = K_y(\alpha')$. This means that the Minkowski spaces $(T^*_x M, K_x)$ and $(T^*_y M, K_y)$ are linearly isometric for every $x, y \in M$, q.e.d.

Another interesting property of BC spaces is as follows.

The connection coefficients $H^i_{jk}(x,p) = H^i_{jk}(x)$ define a symmetric linear connection ∇ on M and it happens that this is *metrizable*, that is, there exists on M a Riemannian metric h such that ∇ is the Levi–Civita connection associated to it. This h is not unique.

We prove this fact by adapting an idea of Z.I. Szabó [6]. The duality with Finsler spaces is not used.

Theorem 3.3. *Let* $C^n = (M, K)$ *be a* BC *space with* M *connected and* ∇ *the symmetric linear connection on* M *of local coefficients* $H^i_{jk}(x,p) = H^i_{jk}(x)$*. Then there exists a Riemannian metric* h *on* M *such that* ∇ *is the Levi–Civita connection of it.*

Proof. Let be the Minkowski space $(T^*_{x_0}M, K_{x_0})$ for a fixed $x_0 \in M$. Then $S_{x_0} = \{\omega \mid K_{x_0}(\omega) = 1\}$ is a compact subset of $T^*_{x_0}M$. Let G be the group of all linear isomorphisms of $T^*_{x_0}M$ that preserve S_{x_0}. This G is a compact Lie group. It contains as a subgroup the holonomy group H_{x_0} defined by $(H^i_{jk}(x))$ according to Lemma 3.1. In general, H_{x_0} is not compact.

Let $<,>$ be *any* inner product in $T^*_{x_0}M$. Define a new inner product on $T^*_{x_0}M$ by

$$h_{x_0}(\varphi, \omega) = \frac{1}{\mathrm{vol}(G)} \int_G < a\varphi, a\omega > \mu_G, \ \varphi, \omega \in T^*_{x_0}M, \tag{3.1}$$

for $a \in G$, where μ_G denotes the bi–invariant Haar measure on G. It results $h_{x_0}(b\varphi, b\omega) = h_{x_0}(\varphi, \omega)$ for every $b \in G$ (from the properties of μ_G), that is h_{x_0} is G-invariant. In particular, h_{x_0} is H_{x_0}-invariant.

Let now any $x \in M$ and a curve $c : t \to c(t)$ joining x with x_0, $c(0) = x$, $c(1) = x_0$. Denote by $P_c : T_x^*M \to T_{x_0}^*M$ the parallel transport of covectors defined by $H^i_{jk}(x)$. For every $\varphi \in T_x^*M$, $P_c(\varphi) = \omega(1) \in T_{x_0}^*M$, where $\omega = (\omega_i)$ is the unique solution of the system of linear differential equations

$$\frac{d\omega_i}{dt} - H^i_{jk}\omega_j \frac{dx^k}{dt} = 0, \text{ with } \omega(0) = \varphi. \tag{3.2}$$

In the proof of Theorem 3.2 we have seen that P_c is a linear isometry of Minkowski spaces. We define an inner product on T_x^*M by

$$h_x(\varphi, \psi) = h_{x_0}(P_c\varphi, P_c\psi), \ \varphi, \psi \in T_{x_0}^*M. \tag{3.3}$$

Lemma 3.2. *h_x does not depend on the curve c.*

Indeed, if $\widetilde{c}$ is another curve joining x and x_0, denote by c_- the reverse of c and consider the loop $\widetilde{c} \circ c_-$. Then $P_{\widetilde{c}\circ c_-} \in H_{x_0}$ and from the H_{x_0}-invariance of h_{x_0}, that is, $h_{x_0}(P_{\widetilde{c}\circ c_-}\varphi, \ P_{\widetilde{c}\circ c_-}\psi) = h_{x_0}(\varphi, \psi)$ we get $h_{x_0}(P_{\widetilde{c}}\varphi, P_{\widetilde{c}}\psi) = h_{x_0}(P_c\varphi, P_c\psi)$ as we claimed.

The mapping $x \to h_x : T_x^*M \times T_x^*M \to R$ is smooth since P_c smoothly depends on x, according to a general result regarding the dependence of solution of system of differential equations by initial data. Thus we have constructed a Riemannian metric h in the cotangent bundle of M.

The connection coefficients $(H^i_{jk}(x))$ define a linear connection ∇ in the cotangent bundle as follows:

$$\nabla : \mathcal{X}(M) \times \Gamma(T^*M) \to \Gamma(T^*M), \ (X, \omega) \to \nabla_X \omega = X^k \left(\frac{\partial \omega_i}{\partial x^k} - H^j_{ik}\omega_j \right) dx^i$$

and the operator ∇_X, $X \in \mathcal{X}(M)$, extends to the tensorial algebra of the cotangent bundle. For instance, if we regard h as a section in the vector bundle $L^s_2(T^*M, \mathbb{R})$, then we have

$$(\nabla_X h)(\varphi, \psi) = X(h(\varphi, \psi)) - h(\nabla_X\varphi, \psi) - h(\varphi, \nabla_X\psi). \tag{3.4}$$

Lemma 3.3. $\nabla_X h = 0, \ X \in \mathcal{X}(M)$.

Proof. We choose a basis $(\varphi_i(x))$ in T_x^*M. It suffices to show that $(\nabla_X h)(\varphi_i(x), \varphi_j(x)) = 0$. Let be the vector $X = \left.\frac{dc}{dt}\right|_\circ$ tangent to a curve c starting from $x \in M$ at $t = 0$. We parallel translate $\varphi_i(x)$ along c and we obtain a field of basis $\varphi_i(t)$ along c. The general formula

$$\frac{\nabla h}{dt}(\varphi, w) = \frac{dh(\varphi, \psi)}{dt} - h\left(\frac{\nabla\varphi}{dt}, \psi\right) - h\left(\varphi, \frac{\nabla\psi}{dt}\right),$$

gives

$$\frac{\nabla h}{dt}(\varphi_i(x), \varphi_j(x)) = \left.\frac{dh(\varphi_i, \varphi_j)}{dt}\right|_{t=0}$$

because of $\dfrac{\nabla \varphi_i}{dt} = 0$.

Now we show that $h(\varphi_i(t), \varphi_j(t))$ does not depend on t.

Indeed, $h_{c(t)}(\varphi_i(t), \varphi_j(t)) = h_{x_0}(P_{\varphi_i}, P_{\varphi_j})$, where P is the parallel translation from $T^*_{c(t)}M$ to $T_{x_0}M$. This P may be thought as the composition of a parallel translation P_2 from $T^*_{c(t)}M$ to T^*_xM and of a parallel translation P_1 from T^*_xM to $T^*_{x_0}M$. We have $h_{c(t)}(\varphi_i(t), \varphi_j(t)) = h_{x_0}((P_2 \circ P_1)\varphi_i, (P_2 \circ P_1)\varphi_j) = h_{x_0}(P_1\varphi_i, P_2\varphi_j) = h_x(\varphi_i(x), \varphi_j(x))$. Hence $h_{c(t)}(\varphi_i(t), \varphi_j(t))$ does not depend on t, as we claimed.

This fact ends the proof of Lemma 3.3.

To end the proof of Theorem, we take the covariant part of h as a section in the vector bundle $L^s_2(TM, \mathbb{R})$ and so we get a Riemannian metric on M, denoted with the same letter h. The operator ∇_X acts also on vector fields on M by the rule $\nabla_X Y = X^k \left(\dfrac{\partial Y^i}{\partial x^k} + H^i_{jk} Y^j\right)$ for $Y = Y^i \dfrac{\partial}{\partial x^i}$ and $(X, Y) \to \nabla_X Y$ gives a linear connection on M such that $\nabla_X h = 0$. As ∇ has no torsion, it coincides with the Levi–Civita connection of h, q.e.d.

Remark. An alternative way to prove Lemma 3.3 is to prove first that $\dfrac{\nabla h}{dt}(\varphi, \psi) = \lim\limits_{t \to 0} \dfrac{h(P_c\varphi, P_c\psi) - h(\varphi, \psi)}{t}$, where P_c is the parallel translation from $c(0)$ to $c(t)$.

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Faculty of Mathematics,
University “Al.I. Cuza” Iaşi,
6600, Iaşi, Romania

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FINSLER VECTOR BUNDLES. METRIZABLE CONNECTIONS

BY

M. ANASTASIEI[4]

Dedicated to Prof. Dr. Lajos Tamássy at his 80th anniversary

Abstract

A vector bundle $\xi = (E, \pi, M)$ of rank m is called a Finsler vector bundle if E is endowed with a continuous, positive function F which is smooth on $E\backslash 0$, positively homogeneous of degree 1 in fibre variables and whose Hessian is positive definite. Then the fibres $E_x, x \in M$, of ξ are Minkowski spaces with the Minkowski norm $F(x,)$.

A nonlinear connection N in ξ induces a linear connection in the vertical bundle over E (Berwald connection) and an operator $|_k$ of $h-$covariant derivative. We say that N is compatible with F if $F_{|k} = 0$ and in this case we show that the parallel translations of N preserve the norms $F(x,)$. Next we consider the case when the coefficients of the Berwald connection do not depend of the fibre variables and we prove that the linear connection in ξ defined by these coefficients is metrizable. As a corollary a metrizability condition for any linear connection in the Finsler vector bundle ξ is provided.

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Introduction

The notion of Finsler function can be considered not only for tangent bundles but also for any vector bundle and the notion of Finsler vector bundle is obtained. A vector bundle $\xi = (E, \pi, M)$ of rank m is called a **Finsler vector bundle** if E is endowed with a continuous, positive function F which

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is smooth on $E\backslash 0$, positively homogeneous of degree 1 in fibre variables and whose Hessian is positive definte. Any Riemannian metric in ξ defines a Finsler function and Finsler functions of Randers type can be considered. When M is a paracompact manifold, the vector bundle ξ can be endowed with a nonlinear connection N.This defines a linear connection in the vertical bundle over E called the Berwald connection associated to N. We use it in Section 2 in order to define two kinds of compatibility between F and N that coincide when the Berwald connection does not depend on variables from fibres. In this case the Berwald connection may be thought as a linear connection ∇ in ξ and in Section 3 we show that ∇ is a metrizable connection, that is there exists a Riemannian metric h in ξ such that $\nabla h = 0$. As a corollary we point out a metrizability condition for any linear connection in the Finsler vector bundle ξ. For the problem of metrizability of linear connections we refer to the paper [5], [6] by L. Tamassy as well as to our papers [1] and [2].

1 Finsler vector bundles

Let $\xi = (E, p, M)$, $p : E \to M$, be a vector bundle of rank m. Here M is a smooth i.e. C^∞ manifold of dimension n. The type fibre is $\mathbb{R}^m$ and E is a smooth manifold of dimension $n + m$. The projection p is a smooth submersion. Let $(U, (x^i))$ be a local chart on M and let $\varepsilon_a(x)$, $x \in U$, be a field of local sections of ξ over U. Then every section A of ξ over U takes the form $A = A^a(x)\varepsilon_a(x)$, $x \in U$, and an element $u \in p^{-1}(x) := E_x$ can be written as $u = y^a\varepsilon_a(x)$, $(y^a) \in \mathbb{R}^m$. The indices $i, j, k, ...$ will range over $\{1, 2, ..., n\}$ and the indices $a, b, c, ...$ will take their values in $\{1, 2, ..., m\}$. The convention on summation over repeated indices of the same kind will be used.

The local coordinates on $p^{-1}(U)$ will be (x^i, y^a) and a change of coordinates $(x^i, y^a) \to (\widetilde{x}^i, \widetilde{y}^a)$ on $U \cap \widetilde{U} \neq \emptyset$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \ \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n, \\ &\widetilde{y}^a = M^a_b(x)y^b, \ \operatorname{rank}(M^a_b(x)) = m, \quad \forall x \in U \cap \widetilde{U}. \end{aligned} \tag{1.1}$$

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E, respectively and by $\mathcal{X}(M)$, resp. $\Gamma(E)$, $\mathcal{X}(E)$ the module of sections of the tangent bundle of M, resp. of the bundle ξ and of the tangent bundle of E. On U, the vector fields $(\partial_k := \dfrac{\partial}{\partial x^k})$ provide a local basis for $\mathcal{X}(U)$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ. We take as local basis of $\Gamma(E^*)$ on U_α, the sections $\theta^a : U \to p^{*-1}(U)$, $x \to \theta^a(x) \in E^*_x$ such that $\theta^a(\varepsilon_b(x)) = \delta^a_b$. A section β of $\xi^* = (E^*, p^*, M)$ will take the form $\beta(x) = \beta_a\theta^a$.

Next, we may consider the tensor bundle of type (r, s), denoted as $\mathcal{T}^r_s(E) := E\underbrace{\otimes \cdots \otimes}_{r} E \otimes E^* \underbrace{\otimes \cdots \otimes}_{s} E^*$ over M and its sections. For $g \in \Gamma(E^* \otimes E^*)$ we have the local representation $g = g_{ab}(x)\theta^a \otimes \theta^b$. As $E^* \otimes E^* \cong L_2(E, \mathbb{R})$, we

may regard g as a smooth mapping $x \to g(x) : E_x \times E_x \to \mathbb{R}$ with $g(x)$ a bilinear mapping given by $g(x)(s_a, s_b) = g_{ab}(x)$.

If the mapping $g(x)$ is symmetric i.e. $g_{ab} = g_{ba}$ and positive-definite i.e. $g_{ab}(x)\zeta^a\zeta^b > 0$ for every $0 \neq (\zeta^a) \in \mathbb{R}^m$, one says that g defines a Riemannian metric in the vector bundle ξ.

The sets of sections $\Gamma(T^r_s(E))$ are $\mathcal{F}(M)$-modules for every natural numbers r, s. On the sum $\bigoplus\limits_{r,s} \Gamma(T^r_s(E))$ a tensor product can be defined and one gets a tensorial algebra $\mathcal{T}(E)$. For the vector bundle (TM, τ, M) this reduces to the tensorial algebra of the manifold M.

A vector bundle $\xi = (E, p, M)$ is called **a Finsler vector bundle** if it is endowed with a Finsler function defined as follows.

Definition 1.1. Let $\xi = (E, p, M)$ be a vector bundle of rank m. A *Finsler function* on E is a nonnegative real function F on E with the properties

1) F is smooth on $E \setminus \{(x, 0), x \in M\}$,

2) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,

3) The matrix with the entries $g_{ab}(x, y) = \dfrac{1}{2}\,\dfrac{\partial^2 F^2}{\partial y^a \partial y^b}$ is positive definite.

On E we have the vertical distribution $u \to V_u E = \operatorname{Ker} p_{x,u}$, where p_* denotes the differential of p. This consists of vectors which are tangent to fibres and it is locally spanned by $\left(\dot{\partial}_a := \dfrac{\partial}{\partial y^a}\right)$. We shall regard also the vertical distribution as a vector subbundle $VE := \bigcup\limits_{u \in E} V_u E \to E$ of $TE \to E$. Its sections will be called vertical vector fields of E. The tensorial algebra $\mathcal{T}(VE) = \oplus \mathcal{T}^p_q(VE),\ p, q \in \mathbb{N}$ of this subbundle will be used. Its elements will be indicated by the word "vertical".

A Finsler function F on E induces a Riemannian metric g in the vertical bundle over E, given locally by

$$g(\dot{\partial}_a, \dot{\partial}_b) = g_{ab}(x, y). \tag{1.1}$$

It provides also a set of vertical tensor fields by successively deriving it with respect to (y^a)

$$C_{abc}(x, y) = \frac{1}{4}\dot{\partial}_a\dot{\partial}_b\dot{\partial}_c L,\ D_{abcd}(x, y) = \frac{1}{8}\dot{\partial}_a\dot{\partial}_b\dot{\partial}_c\dot{\partial}_d L,\ \text{etc.} \tag{1.2}$$

The homogeneity of F implies that the functions $g_{ab}(x)$ are positively homogenous of degree 0 in y^a and the components of vertical tensor fields from (1.2) are positively homogeneous in y^a of degree $-1, -2, \dots$ etc. When the Euler theorem on homogeneous functions is applied to F one gets

$$F^2(x, y) = g_{ab}(x, y) y^a y^b. \tag{1.3}$$

If the functions g_{ab} do not depend on y we obtain the simplest example of Finsler function on E. We may put this differently. Let $h_{ab}(x)$ be a

Riemannian metric in the vector bundle ξ. Then F given by $F^2(x, y) = h_{ab}(x)y^a y^b$ is a Finsler function on E. Thus any Riemannian vector bundle is a particular Finsler vector bundle. On using the Riemannian metric $h_{ab}(x)$ as well as the components $\beta_a(x)$ of a section β in ξ^* and assuming that $h^{ab}\beta_a\beta_b < 1$ one may construct a Finsler function of Randers type on E as follows

$$F(x, y) = \sqrt{h_{ab}(x)y^a y^b} + \beta_a(x)y^a. \tag{1.4}$$

If we set $\alpha = \sqrt{h_{ab}(x)y^a y^b}$ and $\beta = \beta_a(x)y^a$ a Finsler function on E can be given as

$$F(x, y) = L(\alpha, \beta). \tag{1.5}$$

for L a homogeneous of degree one function in the both variables.

2 Finsler vector bundles with nonlinear connections

Let $\xi = (E, \pi, M)$ be a Finsler vector bundle of rank m endowed with the Finsler function F.

Definition 2.1 A nonlinear connection N on E is a distribution $N : u \to N_uE$, $u \in E$, on E, which is supplementary to the vertical distribution $u \longrightarrow V_u$ on E.

We take the distribution N as being locally spanned by $\delta_k = \partial_k - N_k^a(x, y)\dot{\partial}_a$. By a change of coordinates (1.1), the condition $\delta_k = \dfrac{\partial\widetilde{x}^i}{\partial x^k}\ \widetilde{\delta}_i$ is equivalent with

$$\widetilde{N}_j^a \partial_k \widetilde{x}^j = M_b^a(x)N_k^b(x, y) - \partial_k(M_b^a(x))y^b \tag{2.1}$$

It is important to notice that from (2.1) it follows that the set of functions $F_{bk}^a(x, y) = \dot{\partial}_b N_k^a(x, y)$ behaves under a change of coordinates (1.1) as the local coefficients of a linear connection in the vertical bundle over ξ, that is

$$\widetilde{F}_{bk}^a(\widetilde{x}(x), \widetilde{y}(x, y)) = M_c^a(x)\widetilde{M}_b^d(\widetilde{x}(x))\frac{\partial x^i}{\partial \widetilde{x}^k}\ F_{di}^c(x, y) - \partial_i(M_c^a(x))\frac{\partial x^i}{\partial \widetilde{x}^k}\ y^c, \tag{2.2}$$

where $\left(\dfrac{\partial x^i}{\partial \widetilde{x}^k}\right)$ is the inverse matrix of $\left(\dfrac{\partial \widetilde{x}^k}{\partial x^j}\right)$ and $(\widetilde{M}_b^d)$ denotes the inverse matrix of (M_c^b).

We should like to construct a linear connection D in the vertical bundle $VE \to E$. In order to do this it suffices to provide $D_{\delta_k}\dot{\partial}_a$ and $D_{\dot{\partial}_a}\dot{\partial}_b$. Using (2.2) we have the possibility

$$D_{\delta_k}\dot{\partial}_a = F_{ak}^b(x, y)\dot{\partial}_b,\ D_{\dot{\partial}_b}\dot{\partial}_c = V_{bc}^a(x, y)\dot{\partial}_a, \tag{2.3°}$$

where necessarily $(V^a_{bc}(x,y))$ behave like the components of a vertical tensor field of type $(1,2)$.

In particular, we may take $V^a_{bc}=0$ and introduce

Definition 2.2. The linear connection D in the vertical bundle $VE \to E$ given by

$$D_{\delta_k}\dot{\partial}_a = F^b_{ak}(x,y)\dot{\partial}_b, \quad D_{\dot{\partial}_a}\dot{\partial}_b = 0, \tag{2.3}$$

is called the *Berwald connection* associated to N.

Definition 2.3. We call the pair (ξ, N) a Berwald bundle if the functions $F^a_{bk}(x,y) = \dot{\partial}_b N^a_b(x,y)$ depend on x only.

When (ξ, N) is a Berwald bundle, the functions $F^a_{bk}(x,y) = F^a_{bk}(x)$ define a linear connection ∇ in ξ by

$$\nabla_{\partial_k}\varepsilon_b = F^a_{bk}(x)\varepsilon_a, \tag{2.4}$$

for (ε_a) a basis of local sections in ξ.

Conversely, if ξ is endowed with a linear connection of local coefficients $\Gamma^a_{bk}(x)$, then the functions

$$N^a_k(x,y) = \Gamma^a_{bk}(x)y^b, \tag{2.5}$$

define by setting $\delta_k = \partial_k - N^a_k(x,y)\dot{\partial}_a$ a nonlinear connection on E such that (ξ, N) becomes a Berwald bundle. In other words, any vector bundle endowed with a linear connection is a Berwald bundle.

We notice that the nonlinear connection (2.5) is positively homogeneous of degree 1 in $y = (y^a)$. This suggests us to confine ourselves to the pairs (ξ, N) with the functions $(N^a_k(x,y))$ positively homogeneous of degree 1 in y. The examples to be given later will fall in this category. This assumption requires to eliminate from E the image of the null section as we shall do in the following.

It is well known that, see R.Miron [4], R. Miron and M. Apostasies [5], the Berwald connection induces a covariant derivative in the tensorial algebra of the vertical bundle. This splits in two operators of covariant derivative. The first one is called h–covariant derivative and is defined on functions and vertical vector fields as follows:

$$f_{|k} = \delta_k f, \;\; X^a_{|k} = \delta_k X^a + F^a_{bk}(x,y)X^b. \tag{2.6}$$

It is extended by usual rules to any vertical tensor field. The second, called the v-covariant derivative, is simply the partial derivative with respect to y

$$f\Big|_a = \dot{\partial}_a f, \;\; X^a\Big|_b = \dot{\partial}_b X^a, \tag{2.7}$$

since we have chosen $V^a_{bc} = 0$.

We use the notation $_{|k}$ and $\Big|_a$ for denoting the h- and v-covariant derivatives of any vertical tensor field.

Lemma 2.1. *Let ξ be endowed with a positively 1-homogeneous nonlinear connection N and $_{|k}$ the h-covariant derivative defined by the Berwald connection associated to it. Then*

$$y^a_{|k} = 0, \tag{2.8}$$

holds.

Proof. $y^a_{|k} = \delta_k y^a + F^a_{bk}(x,y)y^b = F^a_{bk}(x,y)y^b - N^a_k(x,y) = 0$ because of Euler theorem on homogeneous functions.

Lemma 2.2. *Let (ξ, N) be a Berwald bundle. Then for any vertical tensor field T of local coefficients $T^{a_1\ldots a_r}_{b_1\ldots b_s}(x,y)$ we have*

$$T^{a_1\ldots a_r}_{b_1\ldots b_s}{}_{|k}\Big|_a = T^{a_1\ldots a_r}_{b_1\ldots b_s}\Big|_{a|k}. \tag{2.9}$$

Proof. One verifies (2.9) by a direct calculation keeping in mind that the functions $F^a_{bk} = \dot{\partial}_a N^a_k$ do not depend on y.

We recall that in $\xi = (E, p, M)$, E means in fact $E \setminus \{(x,0),\ x \in M\}$.

Definition 2.2. Let (ξ, F) be a Finsler vector bundle endowed with a positively 1-homogeneous nonlinear connection N. We say that N is weakly compatible with F if

$$F_{|k} := \delta_k F = 0. \tag{2.10}$$

In the following $N(N^a_i)$ will denote a positively 1-homogeneous nonlinear connection. Given N we may consider the Berwald connection $(\dot{\partial}_b N^a_i, 0)$ and we may speak about $g_{ab|k}$.

Definition 2.3. Let (ξ, F) be a Finsler vector bundle endowed with a positively 1-homogeneous nonlinear connection N. We say that N is strongly compatible with F if

$$g_{ab|k} = 0. \tag{2.11}$$

The terminology just introduced is explained by

Lemma 2.3. *Let (ξ, F) be a Finsler vector bundle endowed with a positively 1-homogeneous nonlinear connection N. Then $g_{ab|k} = 0$ implies $F_{|k} = 0$. The converse holds if the functions $(\dot{\partial}_b N^a_i)$ depends on x only.*

Proof. We covariantly derive in the equality (1.3) and we get $F^2_{|k} = g_{ab|k}y^a y^b + 2g_{ab}(x,y)y^a y^b_{|k} = 0$ by (2.11) and the Lemma 2.1. For the converse, we covariantly derive in the equality defining g_{ab}. If the functions $(\dot{\partial}_b N^a_i)$ do not depend on y, the Lemma 2.2 applies in order to get

$$g_{ab|k} = \frac{1}{2}\frac{\partial^2(F^2_{|k})}{\partial y^a \partial y^b} = 0,$$

q.e.d.

Let be $c : [0, 1] \to M$, $t \to c(t)$, $t \in [0, 1]$ a smooth curve on E. A section A of ξ along c given as $A(t) = A^a(t)\varepsilon_a$ is said to be *parallel* with respect to the nonlinear connection N if $A_*(\dot{c})$ are horizontal vectors. Here A_* means the differential of the section $A : M \to E$. A direct calculation shows that the section A is parallel along the curve c if and only if in any local chart on M, we have

$$\frac{dA^a}{dt} + N_k^a(c(t), A(t)\ \frac{dx^k}{dt} = 0, \tag{2.12}$$

where $t \to x^k(t))$ are the local equations of the curve c.

For the initial conditions $c(0) = x$ and $A^a(0) = A_0^a$, the system of differential equations (2.12) admits an unique solution $A^a(x(t))$ and if one assigns to $(A_0^a) \in E_x$ the element $A^a(x(1)) \in E_{c(1)=z}$ one obtains an application $P_c : E_x \to E_z$ called *parallel translation* along c, defined by N. We notice that because of the homogeneity of the functions N_i^a the solutions of (2.12) are defined on $[0, 1]$. The application $P_c : E_x \to E_z$ is a bijection and in general is not a linear map since the system (2.12) is not a linear one.

Now if one considers all loops on M in $x \in M$, the corresponding parallel translations as bijections from $E_x \to E_x$ provide a group with respect to their composition, called the helotomy group $\phi(x)$ of N in $x \in M$. This is not a linear group.

Let F_x be the restriction of F to the fibre E_x. We call F-map a bijection $f : (E_x, F_x) \to (E_z, F_z)$ with the property $F_x(u) = F_z(f(u))$ for every $u \in E_x$.

Theorem 2.1. *Let the Finsler vector bundle (ξ, F) be endowed with a nonlinear connection N which is weakly compatible with F. Then all parallel translations of ∇ are F-maps. In particular, the holonomy groups $\phi(x)$, $x \in M$, consists of F-maps.*

Proof. Let $c : [0, 1] \to M$ be a curve joining the points $x = c(0)$ and $z = c(1)$ of M. Consider a parallel section $A(t) := A(c(t))$, $t \in [0, 1]$, of ξ along c. We show that the function $f : t \to F(x(t), A(t))$, $t \in [0, 1]$, is constant. Indeed,

$$\frac{dF(x(t), A(t))}{dt} = (\partial_k)\ \frac{dx^k}{dt} + (\dot{\partial}_a F)\ \frac{dA^a}{dt} \overset{(2.4)}{=} F_{|k}\ \frac{dx^k}{dt} = 0.$$

Consider $A_0 \in E_x$ and $A(t)$ the unique solution of (2.4) with the initial condition A_0. Then $P_c(A_0) = A_1$, where $A_1 = A(1)$ and since f is constant, we get $F_x(A_0) = F_z(A_1) = L_z(F_c(A_0))$, q.e.d.

3 Metrizability of Berwald connection

Let the Finsler vector bundle (ξ, F) be endowed with a nonlinear connection N which is weakly compatible with F and such that (ξ, N) is a Berwald bundle. Then by Theorem 2.1, all parallel translations defined by ∇ are isometries, that is, linear F- maps.

In particular, the elements of $\phi(x)$ are isometries of the Minkowski space (E_x, F_x). And $\phi(x)$ is a subgroup of the $G(I_x)$, the group of all linear isomorphisms which leave invariant the indicatrix I_x.

These facts are basic in the proof of the main result of this section.

Theorem 3.1. *If ξ, F is endowed with a nonlinear connection N which is weakly compatible with F and ξ, N is a Berwald bundle, then the linear connection ∇ is metrizable, that is, there exists a Riemannian metric h in ξ such that $\nabla h = 0$.*

Proof. Let be $x_0 \in M$ and the Minkowski space (E_{x_0}, F_{x_0}). The indicatrix I_x is compact. It follows that the group $G := G(I_x)$ is a compact Lie group. We know that G contains $\phi(x)$ as a Lie subgroup but in general $\phi(x)$ is not compact. Let $< \cdot >$ be an arbitrary inner product in E_{x_0}. Define a new inner product on E_{x_0} by

$$h_{x_0}(u,v) = \frac{1}{\mathrm{Vol}(G)} \int_G < gu, gv > \mu_G, \quad \text{for } g \in G,\ u, v \in E_{x_0},$$

where μ_G denotes the bi–invariant Haar measure on G. It follows that h_{x_0} is G–invariant and, in particular, it is $\phi(x_0)$–invariant, i.e., $h_{x_0}(Pu, Pv) = h_{x_0}(u, v)$ for any $P \in \phi(x_0)$. Now we transfer h_{x_0} to all the points of M. For any point $x \in M$, we consider a curve c joining x with x_0 ($c(0) = x,\ c(1) = x_0$).

Define $h_x(A, B) = h_{x_0}(P_c A, P_c B),\ A, B \in E_x$. The property that h_{x_0} is $\phi(x_0)$–invariant assures that h_x does not depend on the curve c.

The mapping $h : x \longrightarrow h_x : E_x \times E_x \to R$ is smooth since P_c smoothly depends on x by a general result about dependence of solutions of an ordinary differential equation on initial data. Thus a Riemannian metric h in ξ is obtained. The proof is ended with the help of

Lemma 3.1. *Let h be a Riemannian metric in ξ and $t \to c(t)$, $t \in \mathbb{R}$, a curve with $c(0) = x \in M$. Then*

$$\text{(3.1)} \qquad \lim_{t \to 0} \frac{1}{t} \left(h_{c(t)}(P_c A, P_c B) - h_x(A, B) \right) = \left(\nabla_{\dot{c}(0)} h \right)(A, B)(x),$$

where $A, B \in E_x$ and $P_c : E_x \to E_{c(t)}$ is the parallel translation along c.

Indeed, by the definition of h, the term in the left side of (3.1) vanishes.

For the proof of Lemma 3.1 we refer to [1].

Corollary 3.1. *Let Γ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Suppose that E is endowed with a Finsler function F with the associated Finsler metric $g_{ab}(x, y)$. Let $_{|k}$ be the h-covariant derivative operator induced by Γ. If $g_{ab|k} = 0$, then Γ is metrizable.*

Proof. The linear connection Γ induces an h- covariant derivative operator and if $g_{ab|k} = 0$ the Theorem 3.1 applies to get that Γ is metrizable.

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GEOMETRY OF LAGRANGIANS AND SEMISPRAYS ON LIE ALGEBROIDS

BY

M. ANASTASIEI

Abstract

One considers a regular Lagrangian L on the total space of a Lie algebroid and one associates to it a semispray suggested by the form of the Euler-Lagrange equations established by A. Weinstein, [5]. Some properties of this semispray are pointed out.

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Key words: regular Lagrangian, Euler- Lagrange equations, semisprays, Lie algebroids

1 Introduction

In a paper appeared in 1996, [5], Alan Weinstein proposed a Lagrangian formalism for Lie algebroids. This is general enough to include several Lagrangian formalisms as those on tangent bundles, on tangent subbundles and on Lie algebras. He obtains the Euler - Lagrange equations using the Poisson structure on the dual of the given Lie algebroid and the Legendre transformation defined by a regular Lagrangian on it. He also defines a notion of semispray. Later on, E. Martinez, [3], develops a Lagrangian formalism for Lie algebroids that is similar to Klein's formalism, [2]. He mainly uses a vector bundle which replaces the double tangent bundle from the usual case. A notion of semispray appears in this setting, too.

In this paper we are mainly dealing with the notion of semipray in A. Weinstein' sense. In Section 2 we recall necessary facts from the theory of vector bundles and establish the notations following the monograph [4].

Section 3 is devoted to semisprays on Lie algebroids. We give a definition that is a direct generalization of the one used in tangent bundle case and we prove that this is equivalent with the definition given by A. Weinstein, [5]. A local characterization is also provided. Three invariants are associated to any semispray.

In Section 4 we show that any regular Lagrangian on a Lie algebroid induces a semispray. This is done on a direct way: the Euler - Lagrange equations obtained by A. Weinstein suggest the form of the local coefficients of a semispray and by a direct calculation we checked that those coefficients are the appropriate ones. Some examples are pointed out.

2 Vector bundles

Let $\xi = (E, \pi, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n + m$, and $\pi : E \to M$ is a smooth submersion. The fibres $E_x = \pi^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be an atlas on M. A vector bundle atlas is $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}_{\alpha \in A}$ with the bijections $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha(u) = (\pi(u), \varphi_{\alpha,\pi(u)})$, where $\varphi_{\alpha,\pi(u)} : E_{\pi(u)} \to \mathbb{R}^m$ is a bijection. The given atlas on M and a vector bundle atlas provide an atlas $\{(\pi^{-1}(U_\alpha), \Phi_\alpha)\}_{\alpha \in A}$ on E.

Here $\Phi_\alpha : \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\phi_\alpha(u) = (\psi_\alpha(\pi(u)), \varphi_{\alpha,\pi(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^n$ and if (U_β, ψ_β) is another local chart such that $x \in U_\alpha \cap U_\beta \neq \phi$, we set $\psi_\beta(x) = \widetilde{x}^i$ and then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n), \ \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \tag{1.1}$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi_{\alpha,x}^{-1}(e_a) := \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ has the form $u = y^a \varepsilon_a(x)$.

We take (x^i, y^a) as coordinates on E. For the bundle chart $(U_\beta, \varphi_\beta, \mathbb{R}^m)$ we put $\varphi_{\beta,x}^{-1}(e_a) = \widetilde{\varepsilon}_a(x)$ and then $u = \widetilde{y}^a \widetilde{\varepsilon}_a(x)$. If we set $\varepsilon_a(x) = M_a^b(x)\widetilde{\varepsilon}_b$ with $\text{rank}(M_a^b(x)) = m$ it follows that $\widetilde{y}^a = M_b^a(x) y^b$. Thus the mapping $\phi_\beta \circ \phi_\alpha^{-1}$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n), \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{y}^a = M_b^a(x) y^b, \text{rank}(M_b^a(x)) = m. \end{aligned} \tag{1.2}$$

The indices $i, j, k, ...$ and $a, b, c...$ will take the values $1, 2, ...n$ and $1, 2, ...m$, respectively. The Einstein convention on summation will be used.

We denote by $\mathcal{F}(M)$, $\mathcal{F}(E)$ the ring of real functions on M and E respectively, and by $\chi(M)$, respectively $\Gamma(E)$, $\chi(E)$ the module of sections of the tangent bundle of M, respectively of the bundle ξ and of the tangent bundle of E. On U_α, the vector fields $\left(\partial_k := \dfrac{\partial}{\partial x^k}\right)$ provide a local basis for $\chi(U_\alpha)$. The sections $\varepsilon_a : U_a \to \pi^{-1}(U_\alpha)$, $\varepsilon_a(x) = \varphi_{\alpha,x}^{-1}(e_a)$ provide a basis for $\Gamma(\pi^{-1}(U_\alpha))$ and a section $A : U_\alpha \to \pi^{-1}(U_\alpha)$ will take the form $A(x) = A^a(x)\varepsilon_a(x)$, $x \in U_\alpha$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ. We may also consider the tensor bundle $T_s^r(E)$ over E. The set of sections $\Gamma(T_s^r(E))$ are $\mathcal{F}(M)-$modules for any natural numbers r, s. On the sum $\oplus_{r,s}\Gamma(T_s^r(E))$ a tensor product can be defined and one gets a tensor algebra $T(E)$. For the tangent bundle (TM, τ, M) this reduces to the tensor algebra of the manifold M. The tensor algebra of the manifold E could be also involved. Its elements are sections in $\mathcal{T}_s^r(TE)$. The tensorial algebra of E contains the subset of

$d-$tensor fields on E. For a general definition of these tensor fields we refer to [4], Ch. III. Shortly, these tensor fields are defined by components depending on (x^i, y^a) and transforming by a change of coordinates as tensors but with the matrices $\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right)$ and $(M_b^a(x))$ and their inverses, only. Notice that in the law of transformation of a tensor field on E could appear also the matrix $\left(\dfrac{\partial M_b^a(x)}{\partial x^i} y^b\right)$.

A large class of examples is provided by the sections in the vertical bundle over E. We recall that the vertical bundle $VE \to E$ is the union of the fibres $V_u E = \ker \pi_{*,u}$ over $u \in E$, where $\pi_{*,u}$ is the differential of π. A basis of local section of $VE \to E$ is given by $\left(\left.\dfrac{\partial}{\partial y^a}\right|_u\right)$ and its dual is $dy^a|_u$. The local components of any element in $\Gamma(T_s^r(VE))$, transform under a change of coordinates on E with the matrix $(M_b^a(x))$ and its inverse (W_b^a). We call such an element a vertical tensor field.

Now if $L : E \to M$ is a smooth function on E (called usually a Lagrangian) then it is easy to check that functions $\dfrac{\partial L}{\partial y^a}$, $g_{ab} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^a \partial y^b}$, $C_{abc} = \dfrac{1}{2}\dfrac{\partial g_{ab}}{\partial y^c}$ define vertical tensor fields of covariance indicated by the position and number of indices.

3 Semisprays for Lie algebroids

A vector bundle $\xi = (E, \pi, M)$ is called a Lie algebroid if it has the following properties:

1. The space of sections $\Gamma(\xi)$ is endowed with a Lie algebra structure $[,]$;
2. There exists a bundle map $\rho : E \to TM$ (called the *anchor map*) which induces a Lie algebra homomorphism (also denoted by ρ) from $\Gamma(\xi)$ to $\chi(M)$.
3. For any smooth functions f on M and any sections $s_1, s_2 \in \Gamma(\xi)$ the following identity is satisfied

$$[s_1, f s_2] = f[s_1, s_2] + (\rho(s_1)f)s_2.$$

Locally, we set

$$\rho(s_a) = \rho_a^i \frac{\partial}{\partial x^i},\ [\varepsilon_a, \varepsilon_b] = L_{ab}^c s_c, \tag{3.1}$$

A change of local charts implies

$$\widetilde{\rho}_a^i = W_a^b \rho_b^j \frac{\partial \widetilde{x}^i}{\partial x^j}, \tag{3.2}$$

where W_a^b is the inverse of the matrix (M_b^a).

Examples of Lie algebroids: the tangent bundle $\tau : TM \to M$ with ρ =identity, any integrable subbundle of TM with the inclusion as anchor map, TP/G for $P(M,G)$ a $G-$principal bundle, see [5].

For a function f on M one defines its vertical lift f^v on E by $f^v(u) = f(\pi(u))$ and its complete lift f^c on E by $f^c(u) = \rho_a^i y^a \dfrac{\partial f}{\partial x^i}$ for $u = (x,y)$ in E. If $A = A^a(x)\varepsilon_a$ is a section in ξ, the vertical lift A^v is a vector field on E defined by $A^v(x,y) = A^a(x)\dfrac{\partial}{\partial y^a}$ and the complete lift A^c is a vector field on E defined by

$$A^c(x,y) = A^a \rho_a^i \frac{\partial}{\partial x^i} + \left(\rho_b^i \frac{\partial A^a}{\partial x^i} - A^d L_{db}^a\right) y^b \frac{\partial}{\partial y^a}.$$

In particular, $\varepsilon_a^v = \dfrac{\partial}{\partial y^a}, \varepsilon_a^c = \rho_a^i \dfrac{\partial}{\partial x^i} - L_{ab}^d y^b \dfrac{\partial}{\partial y^d}$.

A semispray S for the tangent bundle $\tau : TM \to M$ is a vector field on TM which at the same time is a section in the vector bundle $\tau_* : TTM \to TM$, that is we have $\tau_{TM}(S(u)) = u$ and $\tau_{*,u}(S(u)) = u$, $\forall u \in TM$, where τ_{TM} is the vector bundle projection $TTM \to TM$. It follows that $\tau_{*,u}(S(u)) = \tau_{TM}(S(u))$, $\forall u \in TM$.

This equation suggests the following

Definition 3.1. *Let $\xi = (E, \rho, M)$ be a Lie algebroid with the anchor ρ. A vector field S on E will be called a semispray if*

$$\pi_{*,u}(S(u)) = (\rho \circ \tau_E)(S(u)),\ \forall u \in E \tag{3.3}$$

where $\tau_E : TE \to E$ is the natural projection.

Let $c : I \to M$, $I \subseteq \mathbb{R}$ be a curve on M and let $\widetilde{c} : I \to E$ be any curve on E such that $\pi \circ \widetilde{c} = c$. Denote by $\dot{\widetilde{c}}$ the vector field that is tangent to $\widetilde{c}$.

Definition 3.2. *We say that $\widetilde{c}$ is* **admissible** *if*

$$\pi_*(\dot{\widetilde{c}}) = \rho(\widetilde{c}).$$

In local charts on M and E, we have $c(t) = (x^i(t))$, $\widetilde{c}(t) = (x^i(t), y^a(t))$ and $\dot{\widetilde{c}}(t) = \dfrac{dx^i}{dt}\dfrac{\partial}{\partial x^i} + \dfrac{dy^a}{dt}\dfrac{\partial}{\partial y^a}$, $t \in I$.

It results

Lemma 3.1. *The curve $\widetilde{c}$ is admissible if and only if*

$$\frac{dx^i}{dt}(t) = \rho_a^i(x(t)) y^a(t),\ \forall t \in I. \tag{3.4}$$

Again in local charts, let be $S = X^i \dfrac{\partial}{\partial x^i} + Y^a \dfrac{\partial}{\partial y^a}$ a vector field on E.

This is a semispray if and only if

$$X^i(x,y) = \rho_a^i(x) y^a. \tag{3.5}$$

Thus the coordinates $(Y^a(x,y))$ are not determined. We set for convenience $Y^a = -2G^a$. Furthermore, under a change of coordinates $(x^i, y^u) \to (\widetilde{x}^i, \widetilde{y}^a)$, the coordinates $(X^i), (G^a)$ have to change as follows:

$$\widetilde{X}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}(x) X^j, \tag{3.6}$$

$$\widetilde{G}^a = M^a_b G^b - \frac{1}{2}\frac{\partial M^a_b}{\partial x^i} y^b \rho^i_c y^c. \tag{3.7}$$

Using (3.2) one easily sees that the coordinates $(X^i(x,y))$ given by (3.5) verify (3.6).

Concluding, we have

Theorem 3.1. *A vector field* $S = (\rho^i_a y^a)\dfrac{\partial}{\partial x^i} - 2G^a \dfrac{\partial}{\partial y^a}$ *on* E *is a semispray if and only if the coordinates* (G^a) *transform by* (3.7).

The integral curves of S are given by the system of differential equations

$$\frac{dx^i}{dt} = \rho^i_a(x) y^a, \ \frac{dy^a}{dt} + 2G^a(x,y) = 0. \tag{3.8}$$

It comes out these curves are all admissible. The converse is also true, that is we have

Theorem 3.2. *A vector field on* E *is a semispray if and only if all its integral curves are admissible.*

Remark 3.1. The characterization of a semispray provided by the Theorem 3.2 was taken by A. Weinstein, [5], as definition for a semispray on E.

Remark 3.2. (i) Let us assume that $\rho = 0$. Then the admissible curves are all curves from the fibre E_{x_0}, $x_0(x^i_0) \in M$. The integral curves of a semispray S are given by the equations $\dfrac{dy^a}{dt} + 2G^a(x_0, y) = 0$.

(ii) The system of equations (3.8) is no longer equivalent with a second order differential equations as it happens for TM. Thus the term of "second order differential equations" used sometimes for a semispray is no longer appropriate.

(iii) Let D a distribution on M. We regard it as a subbundle of TM and so we may view it as a Lie algebroid with the natural inclusion as anchor map. Using a local basis on D one can see that the admissible curves are those that are tangent to the distribution D. For details we refer to [1].

Let $\widehat{S}$ be another semispray on E. Then $\widehat{S} = (\rho^i_a y^a)\dfrac{\partial}{\partial x^i} - 2\widehat{G}^a \dfrac{\partial}{\partial y^a}$, where the functions $(\widehat{G}^a(x,y))$ have to satisfy (3.7) under a change of coordinates on E. It follows that $\widehat{S} - S = 2(G^a - \widehat{G}^a)\dfrac{\partial}{\partial y^a}$ and the functions $D^a = G^a - \widehat{G}^a$ transform by the rule

$$\widehat{D}^a = M^a_b D^b. \tag{3.9}$$

So we have proved

Theorem 3.3. *Any two semisprays on E differ by a vertical vector field on E.*

A different point of view on semisprays for algebroids was proposed by E.Martinez,[3]. It can be shortly described as follows.

Let $\mathcal{L}^\pi E$ be the subset of $E \times TE$ defined by $\mathcal{L}^\pi E = \{(u,z) | \rho(u) = \pi_*(z)\}$ and denote by $\pi_L : \mathcal{L}^\pi E \longrightarrow E$ the mapping given by $\pi_L(u,z) = \tau_E(z)$. Then $(\mathcal{L}^\pi E, \pi_L, E)$ is a vector bundle over E of rank $2m$. One proves that this vector bundle is also a Lie algebroid.

One associates to a section A of ξ the vertical lift A^V and the complete lift A^C as sections of $\pi_L : \mathcal{L}^\pi E \longrightarrow E$ given by

$$A^V(u) = (0, A^v(u)),\ A^C(u) = (A(\pi(u)),\ A^c(u)),\quad u \in E.$$

If $\{s_a\}$ is a local basis of $\Gamma(E))$, then $\{s_a^V, s_s^C\}$ is a local basis for $\Gamma(\mathcal{L}^\pi E)$.

The vector bundle $(\mathcal{L}^\pi E, \pi_L, E)$ admits a canonical section C called the *Liouville or Euler section* defined by $C(u) = \left(o, y^a \dfrac{\partial}{\partial y^a}\right)$ for $u = y^a \varepsilon_a \in E$. A section J of the vector bundle $\mathcal{L}^\pi E \bigoplus (\mathcal{L}^\pi E)^* \longrightarrow E$ characterized by the conditions $J(A^V) = 0$, $J(A^C) = A^V$, $A \in \Gamma E$ is called the *vertical endomorphism.* We have that $J^2 = 0$. A section S of the vector bundle $(\mathcal{L}^\pi E, \pi_L, E)$ is said to be a semispray if it satisfies the condition $JS = C$. This definition is equivalent with the preceding one. Indeed, in local coordinates if we set $S = A^a \varepsilon_a^C + S^a \varepsilon_a^V$, the condition $JS = C$ gives $A^a = y^a$ and so $S = y^a \left(\rho_a^i \dfrac{\partial}{\partial x^i} - L_{ab}^c y^b \dfrac{\partial}{\partial y^c}\right) + S^a \dfrac{\partial}{\partial y^a} = y^a \rho_a^i \dfrac{\partial}{\partial x^i} + S^a \dfrac{\partial}{\partial y^a}$ since $L_{ab}^c y^a y^b = 0$.

For a semispray on TM, a case when this is equivalent with a system of second order differential equations (SODE), there exists a way to find geometric invariants that to determine, up to a change of coordinates, the solutions of the system.

This way led to a KCC-theory named so as after Kosambi, Cartan and Chern.

The KCC-theory apparently does not work for semisprays on Lie algebroids. However, at least formally we can associate to a semispray $S = (\rho_a^i y^a) \dfrac{\partial}{\partial x^i} - 2G^a(x,y) \dfrac{\partial}{\partial y^a}$, the following invariants:

$$\zeta^a = 2G^a - \frac{\partial G^a}{\partial y^b} y^b, \tag{3.10}$$

$$\Xi^a = \frac{\partial G^a}{\partial y^b} - \frac{\partial G^a}{\partial y^b \partial y^c} y^c, \tag{3.11}$$

$$\Gamma^a = 2G^a - 2\frac{\partial G^a}{\partial y^b} y^b + \frac{\partial G^a}{\partial y^b \partial y^c} y^b y^c. \tag{3.12}$$

Indeed, it is not difficult to check that all these sets of functions define vertical vector fields on E.

To find a complete list of such invariants could be a future task.

4 A semispray derived from a regular Lagrangian

Let $L : E \to R$ be a regular Lagrangian on the Lie algebroid $(E, [,], \rho)$, that is L is a smooth functions such that the matrix with the entries

$$g_{ab}(x,y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^a \partial y^b}, \tag{4.1}$$

is of rank m.

In [5], one associates to L the Euler - Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) = \rho^i_a \frac{\partial L}{\partial x^i} + L^c_{ba} y^b \frac{\partial L}{\partial y^c}, \tag{4.2}$$

for $c(t) = (x^i(t), y^a(t))$ an admissible curve.

Expanding the derivative in (4.2), using (4.1) and (3.4), we may put (4.2) in the form

$$\frac{dy^a}{dt} + 2G^a_L(x,y) = 0, \tag{4.3}$$

with the notation

$$G^a_L = \frac{1}{4} g^{ab}\left(\frac{\partial^2 L}{\partial y^b \partial x^i}\rho^i_c y^c - \rho^i_b \frac{\partial L}{\partial x^j} - L^c_{bd} y^d \frac{\partial L}{\partial y^c}\right). \tag{4.4}$$

We show that the function (G^a_L) verifies (3.7) under a change of coordinates on E.

We set

$$E_a = 4 g_{ab} G^b, \tag{4.5}$$

where

$$E_a = \frac{\partial^2 L}{\partial y^a \partial x^i}\rho^i_b y^b - \rho^i_a \frac{\partial L}{\partial x^i} - L^c_{ba} y^b \frac{\partial L}{\partial y^c}. \tag{4.6}$$

Then we use (3.2) as well as the following equations:

$$\frac{\partial L}{\partial x^i} = \frac{\partial L}{\partial \widetilde{x}^j}\frac{\partial \widetilde{x}^j}{\partial x^i} + \frac{\partial L}{\partial \widetilde{y}^a}\frac{\partial M^a_c}{\partial x^i} y^c$$

$$\frac{\partial^2 L}{\partial y^a \partial x^i} = M^b_a\left(\frac{\partial^2 L}{\partial y^b \partial \widetilde{x}^j}\frac{\partial \widetilde{x}^j}{\partial x^i} + 2\widetilde{g}_{db}\frac{\partial M^d_c}{\partial x^i} y^c\right) + \frac{\partial L}{\partial \widetilde{y}^d}\frac{\partial M^d_a}{\partial x^i}$$

$$L^c_{ab} M^e_c = M^c_a M^d_b \widetilde{L}^e_{cd} + \rho^k_a \frac{\partial M^e_b}{\partial x^k} - \rho^k_b \frac{\partial M^e_a}{\partial x^k}$$

in order to derive

$$E_a = M_a^b \widetilde{E}_b + 2M_a^b \widetilde{g}_{bd} \frac{\partial M_c^d}{\partial x^i} y^c \rho_d^i y^d. \tag{4.7}$$

Using this in (4.5) one shows that $\widetilde{G}_L^a$ is related to G_L^a as in (3.7).

Thus we have proved

Theorem 4.1. *Let L be a regular Lagrangian on the Lie algebroid $(E,[,],\rho)$. Then L defines a semispray $S_L = (\rho_a^i y^a)\frac{\partial}{\partial x^i} - 2G_L^a(x,y)\frac{\partial}{\partial y^a}$, where the function G_L^a are given by (4.4).*

Example 4.1. Let $g_{ab}(x)$ be the coefficients of a Riemannian metric in the Lie algebroid $(E,[,],\rho)$. Then

$$L(x,y) = g_{ab}(x) y^a y^b \tag{4.8}$$

is a regular Lagrangian on E. The semispray associated to it is determined by the functions

$$G^a = \frac{1}{2} g^{ab} \left(\frac{\partial g_{bc}}{\partial x^i} \rho_d^i - \frac{1}{2} \frac{\partial g_{cd}}{\partial x^i} \rho_b^i - L_{db}^e g_{ec} \right) y^c y^d. \tag{4.9}$$

Example 4.2. A more general example is provided by the regular Lagrangians which are homogeneous of degree 2 in (y^a). By the Euler theorem one obtains

$$L(x,y) = g_{ab}(x,y) y^a y^b, \tag{4.10}$$

where $(g_{ab}(x,y))$ are homogeneous functions of degree 0.

As $\frac{\partial}{\partial y^a}$ are homogeneous functions of degree 1 and the derivative with respect to (x^j) does not affect the degree of homogeneity, it results that the coefficients (G^a) from (4.4) are homogeneous of degree 2 in (y^a). This fact is equivalent with $\zeta^a = 0$ and so we have a meaning of the invariant ζ^a. The corresponding semispray is called a spray.

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MECHANICAL SYSTEMS ON LIE ALGEBROIDS

BY

M. ANASTASIEI

Dedicated to the 70th birthday of Professor Ruggero Maria Santilli

1 Introduction

The simplest mathematical model for a mechanical system is made of a Riemannian manifold (M, g) with M a smooth manifold of states $x = (x^i)$ and $g = (g_{ij}(x))$ a Riemannian metric provided by the kinetic energy $\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$ of the system. The difference between kinetic energy and the potential energy defines the Lagrangian L of the system and the solution curves of the Euler-Lagrange equations written for L are the evolution curves of the system. The regular Lagrangian L is living on the tangent manifold TM and thus a new space (phase space) is coming into play.

In many cases the mechanical systems involve external forces that are not of gradient type. These forces are modelled by a covector field $F = (F_i(x))$ or equivalently by a vector field of components $(g^{ij}F_j)$ on the manifold M and then the second Newton's law of dynamics takes the form $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x)$ and it gives also the evolution curves of the system. Thus a mechanical system with external forces is defined as a triple (M, L, F) for L a regular Lagrangian and F a covector(vector) field. The theory of these systems was extensively and clearly presented in an excellent book by R.M. Santilli, [5].

But there exist cases when the external forces depend also on velocity, that is F is living on TM. The corresponding theory was developed by R. Miron and C. Frigioiu, [2] and Munoz-Lecanda M.C. et al., [4].

On the other hand, A. Weinstein constructed in [6] a Lagrangian formalism on a Lie algebroid. A Lie algebroid is a vector bundle (E, π, M) that is endowed with a Lie bracket $[,]$ for its sections and is anchored to the tangent bundle with a bundle morphism $\rho : E \longrightarrow TM$ that induces on sections a Lie algebra homomorphism denoted also by ρ such that for any two sections A, B and any function f on M we have $[A, fB] = f[A, B] + \rho(A)f.B$. The formalism of A. Weinstein contains the Euler - Lagrange equations for a Lagrangian on E. Thus it is open a way for approaching the theory of mechanical systems with external forces (not of gradient type) on Lie algebroids. This is

the aim of this paper. Moreover, for enlarging the applicability of our theory we assume that the external forces depend on the fibre variables. These variables may have various meaning (velocities in tangent bundle case).

Our main result says that if the system is dissipative then its energy is decreasing on the evolution curves. The energy of the system is also used for constructing a Lyapunov function for an equilibrium point of the system. These are presented in Section 4. The preceding sections are devoted to necessary facts from the theory of vector bundles and of Lie algebroids.

2 Vector bundles

Let $\xi = (E, \pi, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n + m$, and $\pi : E \to M$ is a smooth submersion. The fibres $E_x = \pi^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha,\ \psi_\alpha)\}_{\alpha\in A}$ be an atlas on M. A vector bundle atlas is $\{(U_\alpha,\ \varphi_\alpha,\ \mathbb{R}^m)\}_{\alpha\in A}$ with the bijections $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha(u) = (\pi(u), \varphi_{\alpha,\pi(u)}(u))$, where $\varphi_{\alpha,\pi(u)} : E_{\pi(u)} \to \mathbb{R}^m$ is a bijection. The given atlas on M and a vector bundle atlas provide an atlas $\{(\pi^{-1}(U_\alpha),\ \Phi_\alpha)\}_{\alpha\in A}$ on E. Here $\Phi_\alpha : \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\phi_\alpha(u) = (\psi_\alpha(\pi(u)),\ \varphi_{\alpha,\pi(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^n$ and if (U_β, ψ_β) is another local chart such that $x \in U_\alpha \cap U_\beta \neq \phi$, we set $\psi_\beta(x) = (\widetilde{x}^i)$ and then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n),\ \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \tag{2.1}$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi_{\alpha,x}^{-1}(e_a) := \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ has the form $u = y^a \varepsilon_a(x)$.

We take (x^i, y^a) as coordinates on E. For the bundle chart $(U_\beta, \Psi_\beta, \mathbb{R}^m)$ we put $\varphi_{\beta,x}^{-1}(e_a) = \widetilde{\varepsilon}_a(x)$ and then $u = \widetilde{y}^a \widetilde{\varepsilon}_a(x)$. If we set $\varepsilon_a(x) = M_a^b(x)\widetilde{\varepsilon}_b$ with $\operatorname{rank}(M_a^b(x)) = m$ it follows that $\widetilde{y}^a = M_b^a(x) y^b$. Thus $\Phi_\beta \circ \Phi_\alpha^{-1}$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n), \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{y}^a = M_b^a(x) y^b, \operatorname{rank}(M_b^a(x)) = m. \end{aligned} \tag{2.2}$$

The indices $i, j, k, ...a, b, c...$ will take the values $1, 2, ...n$ and $1, 2, ...m$, respectively. The Einstein convention on summation will be used.

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E respectively, and by $\chi(M)$, respectively $\Gamma(E), \chi(E)$ the module of sections of the tangent bundle of M, respectively of the bundle ξ and of the tangent bundle of E. On U_α, the vector fields $\left(\partial_k := \dfrac{\partial}{\partial x^k}\right)$ provide a local basis for $\chi(U_\alpha)$. The sections $\varepsilon_a : U_a \to p^{-1}(U_\alpha)$, $\varepsilon_a(x) = \varphi_{\alpha,x}^{-1}(e_a)$ provide a

basis for $\Gamma(p^{-1}(U_\alpha))$ and a section $A : U_\alpha \to p^{-1}(U_\alpha)$ will take the form $A(x) = A^a(x)\varepsilon_a(x)$, $x \in U_\alpha$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ. We may also consider the tensor bundle $T^r_s(E)$ over E. The set of sections $\Gamma(T^r_s(E))$ are $\mathcal{F}(M)-$modules for any natural numbers r, s. On the sum $\oplus_{r,s}\Gamma(T^r_s(E))$ a tensor product can be defined and one gets a tensor algebra $T(E)$. For the tangent bundle (TM, τ, M) this reduces to the tensor algebra of the manifold M. The tensor algebra of the manifold E could be also involved. Its elements are sections in $\mathcal{T}^r_s(TE)$. The tensorial algebra of E contains the subset of $d-$tensor fields on E. For a general definition of these tensor fields we refer to [3], Ch. III. Shortly, these tensor fields are defined by components depending on (x^i, y^a) and transforming tensorially by a change of coordinates but with the matrices $\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right)$ and $(M^a_b(x))$ and their inverses, only. Notice that in the law of transformation of a tensor field on E could appear also the matrix $\left(\dfrac{\partial M^a_b(x)}{\partial x^i} y^b\right)$.

A large class of examples is provided by the sections in the vertical bundle over E. We recall that the vertical bundle $VE \to E$ is the union of the fibres $V_uE = \ker \pi_{*,u}$ over $u \in E$, where $\pi_{*,u}$ is the differential of π. A basis of local section of $VE \to E$ is given by $\left(\left.\dfrac{\partial}{\partial y^a}\right|_u\right)$ and its dual is $dy^a|_u$. The local components of any element in $\Gamma(T^r_s(VE))$, transform under a change of coordinates on E with the matrix $(M^a_b(x))$ and its inverse (W^a_b). We call such an element a vertical tensor field.

Now if $L : E \to M$ is a smooth function on E (called usually a Lagrangian) then it is easy to check that functions $\dfrac{\partial L}{\partial y^a}$, $g_{ab} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^a \partial y^b}$, $C_{abc} = \dfrac{1}{2}\dfrac{\partial g_{ab}}{\partial y^c}$ define vertical tensor fields of covariance indicated by the position and number of indices.

3 Lagrangians on a Lie algebroid. Associated semispray

A vector bundle $\xi = (E, \pi, M)$ is called a Lie algebroid if it has the following properties:

1. The space of sections $\Gamma(\xi)$ is endowed with a Lie algebra structure $[,]$;
2. There exists a bundle map $\rho : E \to TM$ (called the *anchor map*) which induces a Lie algebra homomorphism (also denoted by ρ) from $\Gamma(\xi)$ to $\chi(M)$.
3. For any smooth functions f on M and any sections $s_1, s_2 \in \Gamma(\xi)$ the following identity is satisfied

$$[s_1, fs_2] = f[s_1, s_2] + (\rho(s_1)f)s_2.$$

Locally, we set

$$\rho(\varepsilon_a) = \rho^i_a \frac{\partial}{\partial x^i},\ [\varepsilon_a, \varepsilon_b] = L^c_{ab}\varepsilon_c, \tag{3.1}$$

A change of local charts implies

$$\widetilde{\rho}^i_a = W^b_a \rho^j_b \frac{\partial \widetilde{x}^i}{\partial x^j}. \tag{3.2}$$

Examples of Lie algebroids: the tangent bundle $\tau : TM \to M$ with ρ =identity, any integrable subbundle of TM with the inclusion as anchor map, TP/G for $P(M, G)$ a $G-$principal bundle, see [6].

Let $L : E \to R$ be a regular Lagrangian on the Lie algebroid $(E, [,], \rho)$, that is L is a smooth functions such that the matrix with the entries

$$g_{ab}(x, y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^a \partial y^b}, \tag{3.3}$$

is of rank m. Let $c : I \to M$, $I \subseteq \mathbb{R}$ be a curve on M and let $\widetilde{c} : I \to E$ be any curve on E such that $\pi \circ \widetilde{c} = c$. Denote by $\dot{\widetilde{c}}$ the vector field that is tangent to $\widetilde{c}$.

Definition 3.1. *We say that* $\widetilde{c}$ *is* **admissible** *if*

$$\pi_*(\dot{\widetilde{c}}) = \rho(\widetilde{c}).$$

In local charts on M and E, we have $c(t) = (x^i(t))$, $\widetilde{c}(t) = (x^i(t), y^a(t))$ and $\dot{\widetilde{c}}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{dy^a}{dt}\frac{\partial}{\partial y^a}$, $t \in I$.

It results

Lemma 3.1. *The curve* $\widetilde{c}$ *is admissible if and only if*

$$\frac{dx^i}{dt}(t) = \rho^i_a(x(t))y^a(t),\ \forall t \in I. \tag{3.4}$$

In [6], one associates to L the Euler - Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) = \rho^i_a \frac{\partial L}{\partial x^i} + L^c_{ba} y^b \frac{\partial L}{\partial y^c}, \tag{3.5}$$

for $c(t) = (x^i(t), y^a(t))$ an admissible curve.

Expanding the derivative, using (3.3) and (3.4), we may put (3.5) in the form

$$\frac{dy^a}{dt} + 2G^a_L(x, y) = 0, \tag{3.6}$$

with the notation

$$G^a_L = \frac{1}{4} g^{ab}\left(\frac{\partial^2 L}{\partial y^b \partial x^i}\rho^i_c y^c - \rho^i_b \frac{\partial L}{\partial x^j} - L^c_{bd} y^d \frac{\partial L}{\partial y^c}\right). \tag{3.7}$$

Let be $S = \rho_a^i(x) y^a \frac{\partial}{\partial x^i} + Y^a \frac{\partial}{\partial y^a}$ a vector field on E, where the coordinates $(Y^a(x,y))$ are not determined. We set for convenience $Y^a = -2G^a$. Furthermore, under a change of coordinates $(x^i, y^u) \to (\widetilde{x}^i, \widetilde{y}^a)$, the coordinates $(X^i(x,y) = \rho_a^i(x) y^a), (G^a)$ have to change as follows

$$\widetilde{X}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}(x) X^j, \tag{3.8}$$

$$\widetilde{G}^a = M_b^a G^b - \frac{1}{2} \frac{\partial M_b^a}{\partial x^i} y^b \rho_c^i y^c. \tag{3.9}$$

Using (3.2) one easily sees that the coordinates $(X^i(x,y))$ verify (3.8).

We say that the vector field S as above is a semispray on E. For more details on semisprays on E we refer to [1].

Now we show that the function (G_L^a) verifies (3.9) under a change of coordinates on E.

We set

$$E_a = 4 g_{ab} G^b, \tag{3.10}$$

where

$$E_a = \frac{\partial^2 L}{\partial y^a \partial x^i} \rho_b^i y^b - \rho_a^i \frac{\partial L}{\partial x^i} - L_{ba}^c y^b \frac{\partial L}{\partial y^c}. \tag{3.11}$$

Then we use (3.2) as well as the following equations:

$$\frac{\partial L}{\partial x^i} = \frac{\partial L}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^j}{\partial x^i} + \frac{\partial L}{\partial \widetilde{y}^a} \frac{\partial M_c^a}{\partial x^i} y^c$$

$$\frac{\partial^2 L}{\partial y^a \partial x^i} = M_a^b \left(\frac{\partial^2 L}{\partial y^b \partial \widetilde{x}^j} \frac{\partial \widetilde{x}^j}{\partial x^i} + 2 \widetilde{g}_{db} \frac{\partial M_c^d}{\partial x^i} y^c \right) + \frac{\partial L}{\partial \widetilde{y}^d} \frac{\partial M_a^d}{\partial x^i}$$

$$L_{ab}^c M_c^e = M_a^c M_b^d \widetilde{L}_{cd}^e + \rho_a^k \frac{\partial M_b^e}{\partial x^k} - \rho_b^k \frac{\partial M_a^e}{\partial x^k}$$

in order to derive

$$E_a = M_a^b \widetilde{E}_b + 2 M_a^b \widetilde{g}_{bd} \frac{\partial M_c^d}{\partial x^i} y^c \rho_d^i y^d. \tag{3.12}$$

Using this in (3.10) one shows that $\widetilde{G}_L^a$ is related to G_L^a as in (3.9).

Thus we have proved

Theorem 3.1. *Let L be a regular Lagrangian on the Lie algebroid $(E, [,], \rho)$. Then L defines a semispray $S_L = (\rho_a^i y^a) \frac{\partial}{\partial x^i} - 2 G_L^a(x,y) \frac{\partial}{\partial y^a}$, where the function G_L^a are given by (3.7).*

Example 3.1. Let $g_{ab}(x)$ be the coefficients of a Riemannian metric in the Lie algebroid $(E, [,], \rho)$. Then

$$L(x, y) = g_{ab}(x) y^a y^b \tag{3.13}$$

is a regular Lagrangian on E. The semispray associated to it is determined by the functions

$$G^a = \frac{1}{2} g^{ab} \left(\frac{\partial g_{bc}}{\partial x^i} \rho_d^i - \frac{1}{2} \frac{\partial g_{cd}}{\partial x^i} \rho_b^i - L_{db}^e g_{ec} \right) y^c y^d. \tag{3.14}$$

Example 3.2. A more general example is provided by the regular Lagrangians which are homogeneous of degree 2 in (y^a). By the Euler theorem one obtains

$$L(x, y) = g_{ab}(x, y) y^a y^b, \tag{3.15}$$

where $(g_{ab}(x, y))$ are homogeneous functions of degree 0.

As $\dfrac{\partial}{\partial y^a}$ are homogeneous functions of degree 1 and the derivative with respect to (x^j) does not affect the degree of homogeneity, it results that the coefficients (G^a) from (3.4) are homogeneous of degree 2 in (y^a). The corresponding semispray is called a spray.

4 Mechanical Lagrangian systems on algebroids

Let $(E, [,], \rho)$ be a Lie algebroid.

Definition 4.1. *A mechanical Lagrangian system with external forces on the Lie algebroid* $(E, [,], \rho)$ *is* $\sum = (E, L, F)$ *with* L *a regular Lagrangian on* E *and* $F = (F_a(x, y))$ *a vertical covector field.*

Let be the functions

$$\mathcal{L}_a := \frac{d}{dt} \left(\frac{\partial L}{\partial y^a} \right) - \rho_a^i \frac{\partial L}{\partial x^j} - L_{ba}^c y^b \frac{\partial L}{\partial y^c} \tag{4.1}$$

defined on admissible curves on E.

Then the equalities $\mathcal{L}_a = 0$ represent the Euler - Lagrange equations associated to L.

We assume that the evolution equations of the system $\sum$ are as follows:

$$\mathcal{L}_a(x(t), y(t)) = F_a(x(t), y(t)), \tag{4.2}$$

for $\widetilde{c}(t) = (x(t), y(t))$ an admissible curve on E.

The equations (4.2) after some arrangements take the form

$$\frac{dy^a}{dt} + 2G^a(x, y) = \frac{1}{2} F^a(x, y), \tag{4.3}$$

where the functions (G^a) are given by (3.7), $F^a = g^{ab} F_b$, and the equations $\dfrac{dx^i}{dt} = \rho_a^i(x) y^a$ hold.

Thus the evolution equations of the system $\sum$ become

$$
\begin{aligned}
\frac{dx^i}{dt} &= \rho^i_a(x)y^a, \\
\frac{dy^a}{dt} &= -2\left(G^a - \frac{1}{4}F^a\right).
\end{aligned}
\tag{4.4}
$$

The solutions of this system may be regarded as the integral curves of a semispray

$$
S^* = \rho^i_a(x)y^a\frac{\partial}{\partial x^i} - 2G^*(x,y)\frac{\partial}{\partial y^a},\ G^{*a} = G^a - \frac{1}{4}F^a.
\tag{4.5}
$$

Indeed, S^* is a semispray because it differs by the semispray S derived from L by a vertical vector field.

Definition 4.2. *We say that the mechanical Lagrangian system $\sum$ is dissipative if $F_a(x,y)y^a \leq 0$ and that it is strictly dissipative if $F_a(x,y)y^a \leq -\alpha y_a y^a$ with $\alpha > 0$ a constant and $y_a = g_{ab}y^b$.*

Theorem 4.1. *Let be the mechanical Lagrangian system $\sum$ with the evolution equations* (4.4)*. If it is dissipative then its energy $E = y^a\dfrac{\partial L}{\partial y^a} - L$ decreases on the curves that are solutions of* (4.4)*. If furthermore it is strictly dissipative its energy is strictly decreasing on the curves solutions of* (4.4)*, assuming that these have no singularities.*

Proof. Let be $\widetilde{c}(t) = (x^i(t), y^a(t))$ a curve that is a solution of (4.4). Along this curve we have

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{dy^a}{dt}\frac{\partial L}{\partial y^a} + y^a\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) - \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \frac{\partial L}{\partial y^a}\frac{dy^a}{dt} = \\
&= y^a\mathcal{L}_a(x,y) = y^aF_a(x,y).
\end{aligned}
$$

The last equality is based on (4.2) and to obtain the previous one the equations

$$
L^c_{ab}y^ay^b = 0,
\tag{4.6}
$$

have been used.

If the system $\sum$ is dissipative we have $\dfrac{dE}{dt} \leq 0$ and if it is strictly dissipative we have $\dfrac{dE}{dt} \leq -\alpha y_a y^a < 0$, q.e.d.

Now, we show that if $\sum$ is dissipative we can associate to it a Lyapunov function.

Let (x^i_0, y^a_0) be an equilibrium point of S^*.

If ρ is injective this has the form $(x^i_0, 0)$ with $G^{*a}(x^i, 0) = 0$, a condition that is verified if S^* is a spray.

Assume that (x^i_0, y^a_0) is a minimum point for the energy E and set $\widetilde{E}(x,y) = E(x,y) - E(x_0, y_0)$.

We have

$$\widetilde{E}(x_0, y_0) = 0,\ \widetilde{E}(x, y) > 0. \tag{4.7}$$

Let us denote by $\mathcal{L}_{S^*}$ the Lie derivative with respect to S^*.

We have: $\mathcal{L}_{S^*}(E) = \rho^i_a y^a \dfrac{\partial E}{\partial x^i} - 2G^a \dfrac{\partial E}{\partial y^a} + \dfrac{1}{2} F^a \dfrac{\partial E}{\partial y^a}$.

But $\dfrac{\partial E}{\partial y^a} = 2g_{ab}y^b := 2y_a$. Hence $\mathcal{L}_{S^*}(E) = y^a E_a 4G^a y_a + y_a F^a$, where E_a was defined in (3.11). Again (4.6) was used.

Looking at the connection between E_a and G^a it comes out that the first two terms in the expression of $\mathcal{L}_{S^*}(E)$ cancel and so we have

$$\mathcal{L}_{S^*}(E) = y_a F^a \leq 0, \tag{4.8}$$

since $\sum$ is dissipative.

Thus the function $\widetilde{E}$ is a Lyapunov function for S^* in the equilibrium point (x^i_0, y^a_0) but we can not conclude that this point is stable.

In order to do so we need to introduce a Riemannian metric on E and to prove that S^* is complete with respect to that metric. For details see [4].

For $E = TM$ endowed with a regular Lagrangian a Sasaki type metric can be considered but that construction does not work except if the algebroid $(E, [,], \rho)$ is endowed with a nonlinear connection.

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A GENERALIZATION OF MYERS THEOREM

BY

M. ANASTASIEI[5]

Dedicated to Academician Radu Miron at his 80th anniversary

Abstract

The Myers theorem extracts some topological properties of a Riemannian manifold (M, g) from the assumptions that its Ricci curvature is uniformly bounded below by a positive constant. The theorem was extended to Finsler manifolds. Proofs of it can be seen in [1], Ch. 7, [3] Ch.7. In 1979, Galloway ([2]) obtains the same topological properties of (M, g) assuming a weaker boundedness hypothesis on the Ricci curvature.

In this paper we show that the version of Myers theorem due to Galloway holds also for Finsler manifolds. So, a positive answer to a problem posed by B. Suceavă in a private communication is provided.

We mention that B. Suceavă proved a Myers type theorem in the spirit of [2] for almost Hermitian manifolds [4].

Our proof is obtained by modifying some points in the proof from [1] and by checking that some facts proved in [2] for Riemannian manifolds hold also for Finsler manifolds.

Mathematics Subject Classification 2000: 53C60.

Key words: Finsler manifolds, Ricci scalar, Myers theorem.

1 Preliminaries

We shall use the notations and the terminology from [1] without comments.

Let (M, F) be a Finsler manifold. The Finsler structure F is a function $F : TM \to [0, \infty)$, $(x, y) \to F(x, y)$ which is C^∞ on the slit tangent bundle $TM\backslash 0$, positively homogeneous in y and whose Hessian matrix $g_{ij} := \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^i}$ is positive-definite at every point of $TM\backslash 0$.

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The Chern connection is a linear connection in the pull-back bundle π^*TM over $TM\backslash 0$, where $\pi : TM \to M$ is the natural projection. It is only h-metrical and it has two curvatures $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$.

Let be y a non zero element of T_xM. Then $g(x,y) = g_{ij}(x,y)dx^i \otimes dx^j$ is an inner product which is used to measure lengths and angles in T_xM.One calls y a flagpole of the flag (a plane in T_xM) spanned by $l = \dfrac{y}{F(x,y)}$, and another unit vector V which is orthogonal to the flagpole.

The flag curvature is then given as

$$K(x,y,l\wedge V) := V^i(l^j R_{jikh} l^h)V^k =: V^i R_{ik} V^k. \tag{1.1}$$

The raising and lowering of indices is made by using g^{ij} and g_{ij}, respectively. Sometimes, the flag curvature is denoted simply $K(l,V)$. If V is not a unit vector, then we have $g_{(x,y)}(V,V)K(l,V) = V^i R_{ik} V^i$. Let $\{l, e_\alpha, \alpha = 1,\dots,n-1\}$ be a g-orthonormal basis for the fiber of π^*TM over the point $(x,y) \in TM\backslash 0$. With respect to it one has $K(x,y,l\wedge e_\alpha) = R_{\alpha\alpha}$. The Ricci scalar denoted by Ric is

$$Ric := \sum_{\alpha=1}^{n-1} K(x,y,l\wedge e_\alpha) = \sum_{\alpha=1}^{n-1} R_{\alpha\alpha}. \tag{1.2}$$

In any basis one gets

$$Ric = g^{ik} R_{ik}. \tag{1.3}$$

The Ricci tensor is defined as follows

$$Ric_{jk} = \frac{1}{2}\frac{\partial^2(F^2 Ric)}{\partial y^j \partial y^k} \tag{1.4}$$

and one shows that

$$Ric = l^j l^k Ric_{jk}. \tag{1.5}$$

Equivalently,

$$Ric(x,y) = \frac{1}{F^2(x,y)}[y^i y^j Ric_{ij}]. \tag{1.6}$$

If (M,F) has constant flag curvature c, then

$$Ric = (n-1)c, \quad Ric_{jk} = (n-1)c g_{jk}. \tag{1.7}$$

Let $\sigma(t), 0 \le t \le L$, be a unit geodesic with velocity field T. One abbreviates $g_{(\sigma,T)}$ by g_T.

For a vector field $W(t) := W^i(t)\dfrac{\partial}{\partial x^i}$ along σ, the expression,

$$D_T W = \left[\frac{dW^i}{dt} + W^j T^k(\Gamma^i_{jk}(G,T))\right]\frac{\partial}{\partial x^i} \tag{1.8}$$

is called covariant derivative with reference vector T. The formula 1.8 can be stated for any curve but for geodesics one has

$$\frac{d}{dt}g_T(V,W) = g_T(D_TV,W) + g_T(V,D_TW) \tag{1.9}$$

for any vector fields V,W along σ.

Note that (1.9) holds for any curve if V or W is proportional to T.

The constant speed geodesics are solutions of $D_TT = 0$, with reference vector T.

One says that W is parallel long σ if D_TW, with reference vector T. Parallel transport (with reference vector T) one defines on the standard way. By (1.9) the parallel transport preserves g_T-lengths and angles.

For two continuous and piecewise C^∞ vector fields V and W along σ the index form is

$$I(V,W) = \int_0^L [g_T(D_TV,D_TW) - g_T(R(V,T)T,W)]dt. \tag{1.10}$$

Here all D_T are calculated with reference vector T and

$$R(V,T)T := (T^jR^i_{jkh}T^h)V^k\frac{\partial}{\partial x^i}$$

is evaluated at the point (σ,T).

The index form is bilinear and symmetric. We quote from [1] the following facts

Proposition 1.1 [1, p. 174] *Let $\sigma(t) = \exp_p(tT)$, $0 \le t \le r$ be a constant speed geodesic from $p = \sigma(0)$ to $q = \sigma(r)$.*

The following five statements are mutually equivalent:

(a) The point q is not conjugate to p along σ.

(b) Any Jacobi field that vanishes as both points p and q must be identically zero along σ.

(c) Take the variation field of any variation of σ by geodesics. If it vanishes at p and q, then it must be identically zero along σ.

(d) Given any $v \in T_pM$ and $w \in T_qM$, there exits a unique Jacobi field J along σ that equals v at p and w at q.

(e) The derivative $\exp_{p*}$ of the exponent map $\exp_p$ is nonsingular at the location rT in T_pM.

Proposition 1.2 [1, p. 182] *Let $\sigma(t), 0 \le t \le r$ be a geodesic in a Finsler manifold (M,F). Suppose no point $\sigma(t), 0 < t \le r$ is conjugate to $p := \sigma(0)$. Let W be any piecewise C^∞ vector field along σ and let J denote the unique Jacobi field along σ that has the same boundary values as W. That is, $J(0) = W(0)$ and $J(r) = W(r)$. Then*

$$I(W,W) > I(J,J). \tag{1.11}$$

Equality holds if and only if W is actually a Jacobi field, in which case the said J coincides with W.

We close this Section by quoting, for the sake of comparison, the Bonnet-Myers theorem from [1], p. 194:

Let (M, F) be a forward geodesically complete connected Finsler manifold of dimension n. Suppose its Ricci scalar has the following uniform positive lower bound

$$Ric \geq (n-1)\lambda > 0.$$

Equivalently, suppose its Ricci tensor satisfies $y^i y^i Ric_{ij}(x, y) \geq (n-1)\lambda F^2(x, y)$ with $\lambda > 0$. Then:

(1) *Along every geodesic the distance between any two successive conjugate points is at most $\dfrac{\pi}{\sqrt{\lambda}}$. In other words, every geodesic with length $\dfrac{\pi}{\sqrt{\lambda}}$ or longer must contain conjugate points.*

(2) *The diameter of M is at most $\dfrac{\pi}{\sqrt{\lambda}}$.*

(3) *M is in fact compact.*

(4) *The fundamental group $\pi(M, x)$ is finite.*

2 A generalization of Bonnet - Myers theorem

Looking over the proof of Bonnet-Myers theorem given in [1], p. 194-198 it comes out that essential is a proof of its first statement.

Thus we give a more general form of this statement as follows:

Lemma 1. *Let $\sigma(t)$, $0 \leq t \leq L$ be a unit speed geodesic with velocity field T. If*

$$Ric(T, T) \geq a + \frac{df}{dt}, \; for \;\; a \;\; constant\, a > 0 \tag{2.1}$$

and some function f with $|f(t)| \leq C, C \geq 0$, and

$$L \geq \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)}), \tag{2.2}$$

then σ must contain conjugate points.

Remarks.

(i) For $c = 0$ and $a = (n-1)\lambda$, Lemma 2.1 reduces to the assertion (1) of the Bonnet-Myers theorem.

(ii) The condition (2.1) on Ricci allows and negative values of $Ric(T, T)$ along σ.

Proof. Using the parallel transport with reference vector T one construct a moving frame $\{e_i(t)\}$ along σ such that

(i) Each e_i is parallel along σ, that is $D_T e_i = 0$,

(ii) $\{e_i(t)\}$ is a g_T-ortonormal frame,

(iii) $e_n = T$.

Define $W_\alpha(t) = f_\alpha(t) e_\alpha(t)$ for some smooth functions f_α, $\alpha = 1, 2, ...,$ $n - 1$.

Fix a positive $r \le L$ and consider the index from I for $\sigma(t), 0 \le t \le r$. By (1.10) we have

$$I(W_\alpha, W_\alpha) = \int_0^r [\|D_T W_\alpha\|^2 - \|W_\alpha\| K(T, W_\alpha)] dt,$$

where the abbreviation $\|V\| := g_T(V, V)$ was used and $K(T, W_\alpha)$ is the flag curvature evaluated at the point $(\sigma(t), T) \in TM\backslash 0$.

As $D_T W_\alpha = \dfrac{df_\alpha}{dt} e_\alpha$, it results $\|D_T W_\alpha|^2 = |f_\alpha(t)|^2$. It is known that the flag curvature does not depend on vectors spanning the flag. Thus we have $K(T, W_\alpha) = K(T, e_\alpha)$.

Using these facts, $I(W_\alpha, W_\alpha)$ takes the form

$$I(W_\alpha, W_\alpha) = \int_0^r \left[\left(\frac{df_\alpha}{dt} \right)^2 - f_\alpha^2 K(T, e_\alpha) \right] dt.$$

We take $f_\alpha(t) = \sin \dfrac{\pi t}{r}$ and we get

$$I(W_\alpha, W_\alpha) = \frac{\pi^2}{2r} - \int_0^L \sin^2 \frac{\pi t}{r} K(T, e_\alpha) dt.$$

Summing over α one obtains

$$\sum_\alpha I(W_\alpha, W_\alpha) = (n-1)\frac{\pi^2}{2r} - \int_0^r Ric(T, T) \sin^2 \frac{\pi t}{r} dt.$$

By hypotheses, $-Ric(T, T) \le -a - \dfrac{df}{dt}$. Hence

$$\sum_\alpha I(W_\alpha, W_\alpha) \le (n-1)\frac{\pi^2}{2r} - \int_0^r \left(a + \frac{df}{dt} \right) \sin^2 \frac{\pi t}{r} dt.$$

An integration by parts gives first

$$\sum_\alpha I(W_\alpha, W_\alpha) \le (n-1)\frac{\pi^2}{2r} - \frac{ar}{2} + \frac{\pi}{r} \int_0^r f(t) \sin \frac{2\pi t}{r} dt,$$

and then using $|\sin u| \le 1$ and $\|f(t)| \le c$, one finds

$$\sum_\alpha I(W_\alpha, W_\alpha) \le (n+1)\frac{\pi^2}{2r} - \frac{ar}{2} + \pi c$$

and we have $\sum_\alpha I(W_\alpha, W_\alpha) \le o$ if $r \ge \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$ an inequality that holds for $r = L$ by hypothesis. It follows that some $I(W_\alpha, W_\alpha)$ must be nonpositive and let denote that W_α by W.

We proceed by contradiction. Suppose that $\sigma(t), 0 \leq t \leq r = \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$ contains no conjugate points.

By Proposition 1.1, the vector field W, with $W(0) = W(r) = 0$, can not be a Jacobi field since is nowhere zero on $(0, r)$. And by the same Proposition 1.1 the unique Jacobi field which vanishes at the endpoints of $\sigma(t)$, $0 \leq t \leq r$ is identically zero field. By Proposition 1.2 we have $0 = I(J, J) < I(W, W) \leq 0$, which is a contradiction and lemma is proved. In combination with Theorem 7.5.1 from [1], Lemma 1 tell us that the said geodesic σ minimizes arc length among "nearly" piecewise C^∞ curves from $p = \sigma(0)$ to $q = \sigma(r)$, $r = \dfrac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$. The following two consequences of this Lemma cover the content of the Bonnet-Myers theorem.

Theorem 1. *Let (M, F) be a forward geodesically complete connected Finsler manifold. Suppose there exists constants $a > 0$ and $c \geq 0$ such that for every pair of points in M and minimal geodesic σ joining those points having unit tangent vector T, the Ricci curvature satisfies*

$$Ric(T, T) \geq a + \frac{df}{dt} \text{ along } \sigma$$

where f is some function of arclength t satisfying $|f(t)| \leq c$ along σ. Then M is compact and its diam$(M) \leq \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$.

Proof. Since M is forward geodesically complete, by the Hopf-Rinow theorem any pair of points in M can be joined by a minimal geodesic. By Lemma 1, such a geodesic must have the length less than or equal with $\dfrac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$. Thus diam $(M) \leq \dfrac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$ and so M is bounded. Using again the Hopf-Rinow theorem one deduces that M is compact. $\square$

Theorem 2. *Let (M, F) be a forward geodesically complete connected Finsler manifold. Suppose there exist constants $a > 0$ and $c \geq 0$ such that for every pair of points in M (not necessarily distinct) and geodesic σ with unit tangent vector T joining these points, the Ricci curvature satisfies (2.1) where f is some function of the arclength t satisfying $|f(t)| \leq c$ along σ. Then the universal covering manifold of M is compact, with diameter bounded by $\dfrac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$, and hence the fundamental group of M is finite.*

Proof. Let $\widetilde{M}$ be the universal covering manifold of M with the universal covering map $p : \widetilde{M} \to M$. In [1] p. 197 one proves that p endows $\widetilde{M}$ with the same local geometry as M. Repeating word by word the proof of Theorem 1.3 from [2] it comes out that $\widetilde{M}$ satisfies the hypothesis of Theorem 2.1, hence it is compact. It follows its closed subset $p^{-1}(x)$ is compact and being discrete is finite. Since $\pi_1(M, x)$ is bijective with $p^{-1}(x)$ it is itself finite. $\square$

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SEMISPRAYS ON LIE ALGEBROIDS. APPLICATIONS*

BY

M. ANASTASIEI[6]

Dedicated to Prof. Dr. Tomoaki Kawaguchi at his 70th anniversary

Introduction

In any Lagrangian formalism for Lie algebroids (see A. Weinstein, [9][7], E. Martinez, [5]) , the notion of semispray on a Lie algebroid has a central place. If one looks at various definitions of a semispray on a Lie algebroid (see M. Anastasiei, [1]) it comes out that in defining a semispray the anchor map only is used. In the other words, as it will be shown in this paper (Section 2) the notion of semispray can be considered also on the anchored vector bundles. Moreover, we will show in Section 3 that the set of the anchored vector bundle is the largest with this property. Of course, this set includes the set of Lie algebroids and on a Lie algebroid the assertion that any regular Lagrangian on it induces a semispray holds as in the tangent bundle case. We will prove it in Section 4 (see also M. Anastasiei, [1]). We close the paper with an application of semisprays to the mechanical systems on a Lie algebroid (see M. Anastasiei, [3]). The first Section is devoted to some preliminaries on vector bundles.

1 Preliminaries on vector bundles

Let $\xi = (E, \pi, M)$ be a vector bundle of rank m. Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n + m$, and $\pi : E \to M$ is a smooth submersion. The fibres $E_x = \pi^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be an atlas on M. A vector bundle atlas is $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}_{\alpha\in A}$ with the bijections $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha(u) = (\pi(u), \varphi_{\alpha,\pi(u)})$, where $\varphi_{\alpha,\pi(u)} : E_{\pi(u)} \to \mathbb{R}^m$ is a bijection. The given

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[7]Numbers in brackets refer to the references at the end of the paper

atlas on M and a vector bundle atlas provide an atlas $\{(\pi^{-1}(U_\alpha),\ \Phi_\alpha)\}_{\alpha\in A}$ on E.

Here $\Phi_\alpha : \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\phi_\alpha(u) = (\psi_\alpha(\pi(u)),\ \varphi_{\alpha,\pi(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^n$ and if (U_β, ψ_β) is another local chart such that $x \in U_\alpha \cap U_\beta \neq \phi$, we set $\psi_\beta(x) = \widetilde{x}^i$ and then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n),\ \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n. \tag{1.1}$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi_{\alpha,x}^{-1}(e_a) := \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ has the form $u = y^a \varepsilon_a(x)$.

We take (x^i, y^a) as coordinates on E. For the bundle chart $(U_\beta, \varphi_\beta, \mathbb{R}^m)$ we put $\varphi_{\beta,x}^{-1}(e_a) = \widetilde{\varepsilon}_a(x)$ and then $u = \widetilde{y}^a \widetilde{\varepsilon}_a(x)$. If we set $\varepsilon_a(x) = M_a^b(x)\widetilde{\varepsilon}_b$ with $\text{rank}(M_a^b(x)) = m$ it follows that $\widetilde{y}^a = M_b^a(x) y^b$. Thus $\phi_\beta \circ \phi_\alpha^{-1}$ has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, \cdots, x^n), \text{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{y}^a = M_b^a(x) y^b, \text{rank}(M_b^a(x)) = m. \end{aligned} \tag{1.2}$$

The indices $i, j, k, ...$ and $a, b, c...$ will take the values $1, 2, ...n$ and $1, 2, ...m$, respectively. The Einstein convention on summation will be used.

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E respectively, and by $\chi(M)$, respectively $\Gamma(E), \chi(E)$ the module of sections of the tangent bundle of M, respectively of the bundle ξ and of the tangent bundle of E. On U_α, the vector fields $\left(\partial_k := \dfrac{\partial}{\partial x^k}\right)$ provide a local basis for $\chi(U_\alpha)$. The sections $\varepsilon_a : U_a \to \pi^{-1}(U_\alpha)$, $\varepsilon_a(x) = \varphi_{\alpha,x}^{-1}(e_a)$ provide a basis for $\Gamma(\pi^{-1}(U_\alpha))$ and a section $A : U_\alpha \to \pi^{-1}(U_\alpha)$ will take the form $A(x) = A^a(x)\varepsilon_a(x)$, $x \in U_\alpha$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ. We may also consider the tensor bundle $T_s^r(E)$ over E. The set of sections $\Gamma(T_s^r(E))$ are $\mathcal{F}(M)-$modules for any natural numbers r, s. On the sum $\oplus_{r,s}\Gamma(T_s^r(E))$ a tensor product can be defined and one gets a tensor algebra $T(E)$. For the tangent bundle (TM, τ, M) this reduces to the tensor algebra of the manifold M. The tensor algebra of the manifold E could be also involved. Its elements are sections in $\mathcal{T}_s^r(TE)$. The tensorial algebra of E contains the subset of $d-$tensor fields on E. For a general definition of these tensor fields we refer to [6], Ch. III. Shortly, these tensor fields are defined by components depending on (x^i, y^a) and transforming as tensors by a change of coordinates but with the matrices $\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right)$ and $(M_b^a(x))$ and their inverses, only. Notice that in the law of transformation of a tensor field on E could appear also the matrix $\left(\dfrac{\partial M_b^a(x)}{\partial x^i} y^b\right)$.

A large class of examples is provided by the sections in the vertical bundle over E. We recall that the vertical bundle $VE \to E$ is the union of the fibres

$V_uE = \ker \pi_{*,u}$ over $u \in E$, where $\pi_{*,u}$ is the differential of π. A basis of local section of $VE \to E$ is given by $\left(\left. \frac{\partial}{\partial y^a} \right|_u \right)$ and its dual is $dy^a|_u$. The local components of any element in $\Gamma(T^r_s(VE))$, transform under a change of coordinates on E with the matrix $(M^a_b(x))$ and its inverse (W^a_b). We call such an element a vertical tensor field.

2 Semisprays for anchored vector bundles

A vector bundle $\xi = (E, \pi, M)$ is called anchored (with the tangent bundle TM) if there exists a v.b. morphism $\rho : E \mapsto M$ called the anchor map.

The v.b. morphism ρ induces a $\mathcal{F}(M)$ - module homomorphism from $\Gamma(E) \mapsto \chi(M)$ denoted also by ρ.

Locally, we set

$$\rho(\varepsilon_a) = \rho^i_a \frac{\partial}{\partial x^i}. \tag{2.1}$$

A change of local charts implies

$$\widetilde{\rho}^i_a = W^b_a \rho^j_b \frac{\partial \widetilde{x}^i}{\partial x^j}, \tag{2.2}$$

where W^b_a is the inverse of the matrix (M^a_b).

Examples.

1. A trivial example of anchored v.b. is the tangent bundle itself with the identity mapping as anchor.

2. A less trivial example is a provided by a subbundle of the tangent bundle i.e. a distribution D on M with the inclusion mapping as anchor. Let be $dimD = m < n$ and $(X_1, ..., X_m)$ a base of local sections of D. Then we may write $X_a = X^i_a \frac{\partial}{\partial x^i}$ with $rank(X^i_a) = m$. The anchor is given by

   $$\rho(X_a) = X^i_a \frac{\partial}{\partial x^i}, \tag{2.3}$$

3. Let P be a principal $G-$ bundle of projection p over M. Then TP/G is a vector bundle over M whose sections are the $G-$ invariant vector fields on P. The derivative $p_* : TP \mapsto TM$ passes to a mapping from $TP/G \mapsto TM$ which is the anchor.

We recall that a semispray S for the tangent bundle $\tau : TM \to M$ is a vector field on TM which at the same time is a section in the vector bundle $\tau_* : TTM \to TM$, that is we have $\tau_{TM}(S(u)) = u$ and $\tau_{*,u}(S(u)) = u$, $\forall u \in TM$, where τ_{TM} is the vector bundle projection $TTM \to TM$. It follows that $\tau_{*,u}(S(u)) = \tau_{TM}(S(u))$, $\forall u \in TM$.

This equation suggests the following

Definition 2.1. *Let $\xi = (E, \rho, M)$ be a an anchored v.b. with the anchor ρ. A vector field S on E will be called a semispray if*

$$\pi_{*,u}(S(u)) = (\rho \circ \tau_E)(S(u)),\ \forall u \in E \tag{2.4}$$

where $\tau_E : TE \to E$ is the natural projection.

Let $c : I \to M$, $I \subseteq \mathbb{R}$ be a curve on M and let $\widetilde{c} : I \to E$ be any curve on E such that $\pi \circ \widetilde{c} = c$. Denote by $\dot{\widetilde{c}}$ the vector field that is tangent to $\widetilde{c}$.

Definition 2.2. *We say that $\widetilde{c}$ is* **admissible** *if*

$$\pi_*(\dot{\widetilde{c}}) = \rho(\widetilde{c}).$$

In local charts on M and E, we have $c(t) = (x^i(t))$, $\widetilde{c}(t) = (x^i(t), y^a(t))$ and $\dot{\widetilde{c}}(t) = \dfrac{dx^i}{dt}\dfrac{\partial}{\partial x^i} + \dfrac{dy^a}{dt}\dfrac{\partial}{\partial y^a}$, $t \in I$.

It results

Lemma 2.1. *The curve $\widetilde{c}$ is admissible if and only if*

$$\frac{dx^i}{dt}(t) = \rho^i_a(x(t))y^a(t),\ \forall t \in I. \tag{2.5}$$

Again in local charts, let be $S = X^i\dfrac{\partial}{\partial x^i} + Y^a\dfrac{\partial}{\partial y^a}$ a vector field on E.

This is a semispray if and only if

$$X^i(x, y) = \rho^i_a(x)y^a. \tag{2.6}$$

Thus the coordinates $(Y^a(x, y))$ are not determined. We set for convenience $Y^a = -2G^a$. Furthermore, under a change of coordinates $(x^i, y^u) \to (\widetilde{x}^i, \widetilde{y}^a)$, the coordinates $(X^i), (G^a)$ have to change as follows:

$$\widetilde{X}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}(x)X^j, \tag{2.7}$$

$$\widetilde{G}^a = M^a_b G^b - \frac{1}{2}\frac{\partial M^a_b}{\partial x^i} y^b \rho^i_c y^c. \tag{2.8}$$

Using (2.2) one easily sees that the coordinates $(X^i(x, y))$ given by (2.6) verify (2.7).

Concluding, we have

Theorem 2.1. *A vector field $S = (\rho^i_a y^a)\dfrac{\partial}{\partial x^i} - 2G^a\dfrac{\partial}{\partial y^a}$ on E is a semispray if and only if the coordinates (G^a) transform by (2.8).*

The integral curves of S are given by the system of differential equations

$$\frac{dx^i}{dt} = \rho^i_a(x)y^a,\ \frac{dy^a}{dt} + 2G^a(x, y) = 0. \tag{2.9}$$

It comes out that these curves are all admissible. The converse is also true, that is we have

Theorem 2.2. *A vector field on E is a semispray if and only if all its integral curves are admissible.*

Remark 2.1. The characterization of a semispray provided by the Theorem 3.2 was taken by A. Weinstein, [9], as definition for a semispray on Lie algebroids.

Remark 2.2.

(i) Let us assume that $\rho = 0$. Then the admissible curves are all curves from the fibre E_{x_0}, $x_0(x_0^i) \in M$. The integral curves of a semispray S are given by the equations $\dfrac{dy^a}{dt} + 2G^a(x_0, y) = 0$.

(ii) For a distribution D on M the condition (2.5) tell us that the tangent vector field $\dfrac{dc}{dt} = y^a(t)X_a(c(t)$, that is $\dfrac{dc}{dt}$ is a section in the vector subbundle D. In other words the admissible curves are in this case all the curves that are tangent to the distribution D. See also M. Anastasiei, [4].

Let $\widehat{S}$ be another semispray on E. Then $\widehat{S} = (\rho_a^i y^a)\dfrac{\partial}{\partial x^i} - 2\widehat{G}^a \dfrac{\partial}{\partial y^a}$, where the functions $(\widehat{G}^a(x, y))$ have to satisfy (2.8) under a change of coordinates on E. It follows that $\widehat{S} - S = 2(G^a - \widehat{G}^a)\dfrac{\partial}{\partial y^a}$ and the functions $D^a = G^a - \widehat{G}^a$ transform by the rule

$$\widehat{D}^a = M_b^a D^b. \tag{2.10}$$

By (2.10) we have that $D^a \dfrac{\partial}{\partial y^a}$ is a vertical vector field.

So we have proved

Theorem 2.3. *Any two semisprays on E differ by a vertical vector field on E.*

3 Homogeneous semisprays(sprays)on anchored vector bundles

For every real member $c > 0$ let h_c denote the homothety $E \to E$, given by $u \to cu, u \in E$. A semispray S on E is called a spray if

$$S(h_c(u)) = ch_{c,*}S(u). \tag{H}$$

Locally,$h_c : (x^i, y^a) \mapsto (x^i, cy^a)$ and the condition (H) is equivalent with

$$G^a(x, cy) = c^2 G^a(x, y). \tag{H_0}$$

Let be $C = y^a \dfrac{\partial}{\partial y^a}$ the Liouville vector field on E.

Using the Euler theorem on homogeneous functions one verifies that (H_0) is equivalent with

$$[C, S] = S. \tag{3.1}$$

We notice that if we assume that S is smooth on E the condition (H_0) reduces to the assertion that G^a are homogeneous polynomials of degree 2 in y^a because of

Lemma 3.1 ([8]). *Let V and V' be linear spaces and $f : V \mapsto V'$ a mapping that is at least $r > 0$ times differentiable at $0 \in V$ and positively homogeneous of degree r. Then f is a homogeneous polynomial of degree r.*

When S is smooth only on $E \setminus \{0\}$ the condition (H_0) is in use.

As we have seen till now, given an anchored v.b. we may find in principle a semispray by pointing out a set of functions (G^a) subject to (2.8). If someone tries to define a semispray on any vector bundle it is reasonable to try to define first a spray since this has a simpler form. Thus he will start with a vector field S_0 on E that verifies the condition (H_0).

If $S_0 = X^i(x,y)\dfrac{\partial}{\partial x^i} + Y^a(x,y)\dfrac{\partial}{\partial y^a}$, it will result that $(X^i(x,y))$ are linear functions in y^a, that is $X^i = \rho^i_a(x)y^a$ and $(Y^a(x,y))$ are homogeneous polynomials of degree 2 in y^a. The map $\pi_* \circ S_0$ carries a section $y^a\varepsilon_a$ to $\rho^i_a(x)y^a\dfrac{\partial}{\partial x^i}$ i.e. it defines a morphism $E \mapsto TM$. As $\tau_E \circ S_0 = id_E$ holds, the condition (2.1) is fulfilled, i.e. S_0 is a spray.

Concluding, if one wishes the extension of the notion of semispray to vector bundles, one has to assume that vector bundle is anchored. In the other word, the class of anchored v.b. is the largest in which the notion of semispray can be considered. It contains the class of Lie algebroids.

4 A semispray derived from a Lagrangian on a Lie algebroid

A vector bundle $\xi = (E, \pi, M)$ is called a Lie algebroid if it has the following properties:

1. The space of sections $\Gamma(\xi)$ is endowed with a Lie algebra structure $[,]$;
2. There exists a bundle map $\rho : E \to TM$ (called the *anchor map*) which induces a Lie algebra homomorphism (also denoted by ρ) from $\Gamma(\xi)$ to $\chi(M)$.
3. For any smooth functions f on M and any sections $s_1, s_2 \in \Gamma(\xi)$ the following identity is satisfied

$$[s_1, fs_2] = f[s_1, s_2] + (\rho(s_1)f)s_2.$$

Locally, we set

$$\rho(\varepsilon_a) = \rho^i_a\frac{\partial}{\partial x^i},\ [\varepsilon_a, \varepsilon_b] = L^c_{ab}\varepsilon_c.$$

Let $L : E \to R$ be a regular Lagrangian on the Lie algebroid $(E, [,], \rho)$, that is L is a smooth functions such that the matrix with the entries

$$g_{ab}(x, y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^a \partial y^b}, \tag{4.1}$$

is of rank m.

In [9], one associates to L the Euler - Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) = \rho^i_a \frac{\partial L}{\partial x^i} + L^c_{ba} y^b \frac{\partial L}{\partial y^c}, \tag{4.2}$$

for $c(t) = (x^i(t), y^a(t))$ an admissible curve.

Expanding the derivative, using (4.1) and (3.4), we may put (4.2) in the form

$$\frac{dy^a}{dt} + 2G^a_L(x, y) = 0, \tag{4.3}$$

with the notation

$$G^a_L = \frac{1}{4} g^{ab} \left(\frac{\partial^2 L}{\partial y^b \partial x^i} \rho^i_c y^c - \rho^i_b \frac{\partial L}{\partial x^j} - L^c_{bd} y^d \frac{\partial L}{\partial y^c} \right). \tag{4.4}$$

In [1]we have shown by a direct calculation that the function (G^a_L) verify (2.8) under a change of coordinates.

In the other words we have proved

Theorem 4.1. *Let L be a regular Lagrangian on the Lie algebroid $(E, [,], \rho)$. Then L defines a semispray $S_L = (\rho^i_a y^a)\frac{\partial}{\partial x^i} - 2G^a_L(x, y)\frac{\partial}{\partial y^a}$, where the function G^a_L are given by* (4.4).

Example 4.1. Let $g_{ab}(x)$ be the coefficients of a Riemannian metric in the Lie algebroid $(E, [,], \rho)$. Then

$$L(x, y) = g_{ab}(x) y^a y^b \tag{4.5}$$

is a regular Lagrangian on E. The semispray associated to it is determined by the functions

$$G^a = \frac{1}{2} g^{ab} \left(\frac{\partial g_{bc}}{\partial x^i} \rho^i_d - \frac{1}{2}\frac{\partial g_{cd}}{\partial x^i} \rho^i_b - L^e_{db} g_{ec} \right) y^c y^d. \tag{4.6}$$

Example 4.2. A more general example is provided by the regular Lagrangians which are homogeneous of degree 2 in (y^a). By the Euler theorem one obtains

$$L(x, y) = g_{ab}(x, y) y^a y^b, \tag{4.7}$$

where $(g_{ab}(x, y))$ are homogeneous functions of degree 0.

As $\dfrac{\partial}{\partial y^a}$ are homogeneous functions of degree 1 and the derivative with respect to (x^j) does not affect the degree of homogeneity, it results that the coefficients (G^a) from (4.4) are homogeneous of degree 2 in (y^a). The corresponding semispray is nothing but a spray.

5 Mechanical Lagrangian systems on Lie algebroids

Let $(E, [,], \rho)$ be a Lie algebroid.

Definition 5.1. *A mechanical Lagrangian system with external forces on the Lie algebroid* $(E, [,], \rho)$ *is* $\sum = (E, L, F)$ *with* L *a regular Lagrangian on* E *and* $F = (F_a(x, y))$ *a vertical covector field.*

Let be the functions

$$\mathcal{L}_a := \frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) - \rho_a^i \frac{\partial L}{\partial x^j} - L_{ba}^c y^b \frac{\partial L}{\partial y^c} \tag{5.1}$$

defined on admissible curves on E.

Then the equalities $\mathcal{L}_a = 0$ represent the Euler - Lagrange equations associated to L.

We assume that the evolution equations of the system $\sum$ are as follows:

$$\mathcal{L}_a(x(t), y(t)) = F_a(x(t), y(t)), \tag{5.2}$$

for $\widetilde{c}(t) = (x(t), y(t))$ an admissible curve on E.

The equations (5.2) after some arrangements take the form

$$\frac{dy^a}{dt} + 2G^a(x, y) = \frac{1}{2}F^a(x, y), \tag{5.3}$$

where the functions (G^a) are given by (4.4), $F^a = g^{ab}F_b$, and the equations $\dfrac{dx^i}{dt} = \rho_a^i(x)y^a$ hold.

Thus the evolution equations of the system $\sum$ become

$$\begin{aligned} \frac{dx^i}{dt} &= \rho_a^i(x)y^a, \\ \frac{dy^a}{dt} &= -2\left(G^a - \frac{1}{4}F^a\right). \end{aligned} \tag{5.4}$$

The solutions of this system may be regarded as the integral curves of a semispray

$$S^* = \rho_a^i(x)y^a \frac{\partial}{\partial x^i} - 2G^*(x, y)\frac{\partial}{\partial y^a}, \; G^{*a} = G^a - \frac{1}{4}F^a. \tag{5.5}$$

Indeed, S^* is a semispray because it differs by the semispray S derived from L by a vertical vector field.

Definition 5.2. *We say that the mechanical Lagrangian system* $\sum$ *is dissipative if* $F_a(x, y)y^a \leq 0$ *and that it is strictly dissipative if* $F_a(x, y)y^a \leq -\alpha y_a y^a$ *with* $\alpha > 0$ *a constant and* $y_a = g_{ab}y^b$.

Theorem 5.1. *Let be the mechanical Lagrangian system* $\sum$ *with the evolution equations* (5.4). *If it is dissipative then its energy* $E = y^a \dfrac{\partial L}{\partial y^a} - L$

decreases on the curves that are solutions of (5.4). *If furthermore it is strictly dissipative its energy is strictly decreasing on the curves solutions of* (5.4), *assuming that these have no singularities.*

Proof. Let be $\widetilde{c}(t) = (x^i(t), y^a(t))$ a curve that is a solution of (5.4). Along this curve we have

$$\frac{dE}{dt} = \frac{dy^a}{dt}\frac{\partial L}{\partial y^a} + y^a \frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) - \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \frac{\partial L}{\partial y^a}\frac{dy^a}{dt} =$$

$$= y^a \mathcal{L}_a(x, y) = y^a F_a(x, y).$$

The last equality is based on (5.2) and to obtain the previous one the equations

$$L^c_{ab} y^a y^b = 0, \tag{5.6}$$

have been used.

If the system $\sum$ is dissipative we have $\dfrac{dE}{dt} \leq 0$ and if it is strictly dissipative we have $\dfrac{dE}{dt} \leq -\alpha y_a y^a < 0$, q.e.d.

Now, we show that if $\sum$ is dissipative we can associate to it a Lyapunov function.

Let (x^i_0, y^a_0) be an equilibrium point of S^*.

If ρ is injective this has the form $(x^i_0, 0)$ with $G^{*a}(x^i, 0) = 0$, a condition that is verified if S^* is a spray.

Assume that (x^i_0, y^a_0) is a minimum point for the energy E and set $\widetilde{E}(x, y) = E(x, y) - E(x_0, y_0)$.

We have

$$\widetilde{E}(x_0, y_0) = 0,\ \widetilde{E}(x, y) > 0. \tag{5.7}$$

Let us denote by $\mathcal{L}_{S^*}$ the Lie derivative with respect to S^*.

We have: $\mathcal{L}_{S^*}(E) = \rho^i_a y^a \dfrac{\partial E}{\partial x^i} - 2G^a \dfrac{\partial E}{\partial y^a} + \dfrac{1}{2} F^a \dfrac{\partial E}{\partial y^a}$.

Expanding this and using again (5.6) we get

$$\mathcal{L}_{S^*}(E) = y_a F^a \leq 0, \tag{5.8}$$

since $\sum$ is dissipative.

Thus the function $\widetilde{E}$ is a Lyapunov function for S^* in the equilibrium point (x^i_0, y^a_0) but we can not conclude that this point is stable.

In order to do so we need to introduce a Riemannian metric on E and to prove that S^* is complete with respect to that metric. For details see [7].

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BANACH LIE ALGEBROIDS

BY

M. ANASTASIEI

Abstract

First, we extend the notion of second order differential equations (SODE) on a smooth manifold to anchored Banach vector bundles. Then we define the Banach Lie algebroids as Lie algebroids structures modeled on anchored Banach vector bundles and prove that they form a category.

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Key words: Banach vector bundles, anchor, Second Order Differential Equations, Lie algebroids.

Introduction

Lie algebroids are related to many areas of geometry ([2], [7]) and has recently become an object of extensive studies. See [6] for basic definitions, examples and references. In 1996, Weinstein [8] proposed some applications of the Lie algebroids in Analytical Mechanics. New theoretical developments followed. See the survey [5] by de Leon, Marrero and Martinez about Mechanics on Lie algebroids.

In [1], we gave a construction of a semispray associated to a regular Lagrangian on a Lie algebroid.

In this paper, we consider the notion of Lie algebroid in the category of Banach vector bundles, that is vector bundles over smooth Banach manifolds whose type fibres are Banach spaces. Such a Banach vector bundle over base M is called anchored if there exists a morphism from it to the tangent bundle TM. First, we extend the usual notion of second order differential equations (SODE) to anchored Banach vector bundles and we show that if a Banach vector bundle admits a homogeneous SODE it is necessarily anchored. Then we define the Banach Lie algebroids as Lie algebroid structures modeled on anchored Banach vector bundles. In our setting only one from three equivalent definitions of a morphism of Lie algebroids is working. Using it we show that the Banach Lie algebroids form a category.

1 Anchored Banach vector bundles

Let M be a smooth i.e. C^∞ Banach manifold modeled on a Banach space $\mathbf{M}$ and let $\pi : E \to M$ be a Banach vector bundle whose type fiber is a Banach space $\mathbf{E}$. We denote by $\tau : TM \to M$ the tangent bundle of M.

Definition 1.1 *We say that the vector bundle $\pi : E \to M$ is an anchored vector bundle if there exists a vector bundle morphism $\rho : E \to TM$. The morphism ρ will be called the anchor map.*

Let $\mathcal{F}(M)$ be the ring of smooth real functions on M. We denote by $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections in the vector bundle (E, π, M) and by $\mathcal{X}(M)$ the module of smooth sections in the tangent bundle of M (vector fields on M).

The vector bundle morphism ρ induces an $\mathcal{F}(M)$-module morphism which will be denoted also by $\rho : \Gamma(E) \to \mathcal{X}(M)$, $\rho(s)(x) = \rho(s(x))$, $x \in M, s \in \Gamma(E)$.

Let $\{(U, \varphi), (V, \psi), \ldots\}$ be an atlas on M. Restricting U, V if necessary we may choose a vector bundle atlas $\{(\pi^{-1}(U), \overline{\varphi}), (\pi^{-1}(V), \overline{\psi}), \ldots\}$ with $\overline{\varphi} : \pi^{-1}(U) \to U \times \mathbb{E}$ given by $\overline{\varphi}(u) = (\pi(u), \overline{\varphi}_{\pi(u)})$, where $\overline{\varphi}_{\pi(u)} : E_{\pi(u)} \to \mathbb{E}$ is a toplinear isomorphism. Here $E_{\pi(u)}$ is the fiber of (E, π, M) in $u \in E$. The given atlas on M together with a vector bundle atlas induce a smooth atlas $\{(\pi^{-1}(U), \phi), (\pi^{-1}(U), \psi), \ldots\}$ on E such that E becomes a Banach manifold modeled on the Banach space $\mathbb{M} \times \mathbb{E}$. The map $\phi : \pi^{-1}(U) \to \varphi(U) \times \mathbb{E}$ is given by

$$\phi(u) = (\varphi(\pi(u)), \overline{\varphi}_{\pi(u)}(u)), \quad u \in E.$$

For a section $s : U \to \pi^{-1}(U)$, its local representation $\phi \circ s \circ \varphi^{-1} : \varphi(U) \to \varphi(U) \times \mathbb{E}$ given by $(\phi \circ s \circ \varphi^{-1})(\varphi(x)) = (\varphi\pi(s(x)), \overline{\varphi}_{\pi(s(x))}(s(x)) = (\varphi(x), \overline{\varphi}_x(s(x)))$ is completely determined by the map $s_\varphi : \varphi(U) \to \mathbb{E}$ given by $s_\varphi(\varphi(x)) = \overline{\varphi}_x(s(x))$ which will be called the local representative (shortly l.r.) of s. On $U \cap V$ we may speak also of the l.r. s_ψ of a section $s : U \cup V \to \pi^{-1}(U \cap V)$ given by $s_\psi(\psi(x)) = \overline{\psi}_x(s(x))$. It is clear that we have

$$(1.1) \qquad s_\psi(\psi(x)) = \overline{\psi}_x \circ \overline{\phi}_x^{-1}(s_\varphi(\varphi(x))), \quad x \in U \cup V.$$

For a vector field $X : U \to \tau^{-1}(U)$ we have a l.r. $X_\varphi : \varphi(U) \to \mathbb{M}$ and on $U \cap V$ we have also a l.r. X_ψ and one holds

$$(1.2) \qquad X_\psi(\psi(x)) = d(\psi \circ \varphi^{-1})(\varphi(x))(X_\varphi(\varphi(x))), \quad x \in U \cap V,$$

where d means Frechet differentiation.

Locally, ρ reduces to a morphism $U \times \mathbb{E} \to U \times \mathbb{M}$, $(x, v) \to (x, \rho_U(x)v)$ with $\rho_U(x) \in L(\mathbb{E}, \mathbb{M})$, the space of continuous linear maps from $\mathbb{E}$ to $\mathbb{M}$. We call $\rho_U(x)$ the l.r. of ρ. On overlaps of local charts one easily gets

$$(1.3) \qquad \rho_V(x) \circ \overline{\psi}_x \circ \overline{\varphi}_x^{-1} = d(\psi \circ \varphi^{-1})(\varphi(x)) \circ \rho_U(x), \quad x \in U \cap V$$

Example.

1. The tangent bundle of M is trivially anchored vector bundle with $\rho = I$ (identity).

2. Let A be a tensor field of type $(1,1)$ on M. It is regarded as a section of the bundle of linear mappings $L(TM, TM) \to M$ and also as a morphism $A : TM \to TM$. In other words, A may be thought as an anchor map.

3. Any subbundle of TM is an anchored vector bundle with the anchor the inclusion map in TM.

4. Let $\pi : E \to M$ be only a submersion. The subspaces $V_uE = \pi^{-1}(x), \pi(u) = x$ of TE over E denoted by VE form a subbundle called the vertical subbundle. By Example 3) this is an anchored Banach vector bundle.

The anchored vector bundles over the same base M form a category. The objects are the pairs (E, ρ_E) with ρ_E the anchor of E and a morphism $f : (E, \rho_E) \to (F, \rho_F)$ is a vector bundle morphism $f : E \to F$ which verifies the condition $\rho_F \circ f = \rho_E$.

2 Semisprays in an anchored vector bundle

Let (E, π, M) be an anchored vector bundle with the anchor map ρ and let $\pi_* : TE \to TM$ be the differential (tangent map) of π.

We denote by $\tau_E : TE \to E$ the tangent bundle of E.

Definition 2.1 *A section $S : E \to TE$ will be called a semispray if*

(i) $\tau_E \circ S =$ *identity on E,*

(ii) $\pi_* \circ S = \rho$.

The condition (i) says that S is a vector field on E. The condition (ii) can be written also in the form

$$\pi_{*,u}(S(u)) = \rho(u) = (\rho \circ \tau_E)(S(u)), \quad u \in E.$$

When $E = TM$ and $\rho =$ identity on TM, S is simultaneously a vector field on TM and a section in the vector bundle $\pi_* : TTM \to TM$ i.e. it is a second-order vector field on M in terminology from [3, p.96]. Such a vector field is frequently called a second order differential equation (SODE) on M or a semispray.

As we will see below, in our context S is no more related to a second order differential equation on M and so the corresponding terminology is inadequate.

Let $c : J \to E$ for $\circ \in J \subset \mathbb{R}$ a curve on E. The differential of c is $c_* : J \times \mathbb{R} \to TE$ and using $\imath : J \to J \times \mathbb{R}$, $t \to (t, 1)$, $t \in J$ we set $c'(t) = c_* \circ \imath$. Then in general $\pi \circ c$ is a curve on M and we have that $(\pi \circ c)'(t) = \pi_{*,c(t)} \circ c'(t)$.

Definition 2.2 *A curve c on E will be called admissible if $(\pi \circ c)'(t) = \rho(c(t))$, $\forall t \in J$.*

Locally, if $c : J \to \varphi(U) \times E$, $t \to (x(t), w(t))$ then $\pi \circ c : J \to \varphi(U)$ is $t \to x(t)$, $t \in J$ and it follows that c is an admissible curve if and only if

$$\frac{dx}{dt} = \rho_U(x(t))w(t), \quad t \in J \tag{2.1}$$

Theorem 2.1 *A vector field S on E is a semispray if and only if all its integral curves are admissible curves.*

Proof. Let S be a semispray. A curve $c : J \to E$ is an integral curve of S if $c'(t) = S(c(t))$. It follows $\pi_* \circ c'(t) = (\pi_* \circ S)(c(t))$ or $(\pi \circ c)'(t) = \rho(c(t))$, that is c is an admissible curve. Conversely, let S be a vector field on E whose integral curves are admissible. For every $u \in E$ there exists an unique integral curve $c : J \to E$ of S such that $c(0) = u$ and $c'(0) = S(u)$. We have $\pi_* \circ c'(0) = (\pi_* \circ S)(u)$, $(\pi \circ c)'(0) = (\pi_* \circ S)(u)$ and $\pi_* \circ S = \rho(u)$ since c is admissible. $\square$

We restrict to a local chart (U, φ) on M. Then $TU \simeq \varphi(U) \times \mathbb{M}$, $E_{|U} \simeq \varphi(U) \times \mathbb{E}$ and $TE_{|U} \simeq (\varphi(U) \times \mathbb{E}) \times \mathbb{M} \times \mathbb{E}$.

The l.r. of a vector field on E is $S_\varphi : \varphi(U) \times \mathbb{E} \to \varphi(U) \times \mathbb{E} \times \mathbb{M} \times \mathbb{E}$, $S_\varphi(x, u) = (x, u, S^1_\varphi(x, u), S^2_\varphi(x, u))$. As l.r. of π_* is $\varphi(U) \times \mathbb{E} \times \mathbb{M} \times \mathbb{E} \to \varphi(U) \times \mathbb{M}$, $(x, u, y, v) \to (x, y)$ the condition $\pi_* \circ S = \rho$ translates to $S^1_\varphi(x, u) = (x, \rho_U(x)u)$. We set for convenience $S^2_\varphi(x, u) = -2G_\varphi(x, u)$ and so the l.r. of a semispray for the anchored vector bundle (E, π, M) with the anchor ρ is given as follows:

$$S_\varphi(x, u) = (x, u, \rho_U(x)u, -2G_\varphi(x, u)). \tag{2.2}$$

Let (V, ψ) be another local chart and let us set $h = \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$. Then $h_* : \varphi(U \cup V) \times \mathbb{M} \to \psi(U \cap V) \times \mathbb{M}$ is given by $(x, v) \to (x, dh(x)(v))$, $x \in \varphi(U \cup V), v \in \mathbb{M}$.

Let us denote by $H : \varphi(U \cap V) \times \mathbb{E} \to \psi(U \cap V) \times \mathbb{E}$ the map given by $H(x, u) = (h(x), M(x)u)$, where $M(x) = \overline{\psi}_x \circ \overline{\varphi}_x^{-1} \in L(\mathbb{E}, \mathbb{E})$. Then H_* is locally given as the pair (H, H'): $\varphi(U \cup V) \times \mathbb{E} \times \mathbb{M} \times \mathbb{E} \to \psi(U \cup V) \times \mathbb{E} \times \mathbb{M} \times \mathbb{E}$, where the derivative $H'(x, u)$ is given by the Jacobian matrix operating on the column vector ${}^t(y, w)$ with $y \in \mathbb{M}$ and $w \in \mathbb{E}$. Thus (H, H') takes the form $(x, u, y, v) \to (h(x), M(x)u, h'(x)y, M'(x)(y)(u) + M(x)v)$ with prime being denoted the Frechet derivative.

If S_ψ is l.r. of S in the chart (V, ψ), necessarily we have $(H, H') \circ S_\varphi = S_\psi$ with $S_\psi(x, u) = (h(x), M(x)u, \rho_U(h(x))M(x)u, -2G_\psi(h(x), M(x)u))$.

Computing $(H, H') \circ S_\varphi$ and identifying with S_ψ one finds

$$\begin{aligned} \rho_V(h(x))M(x)(u) &= h'(x)\rho_U(x)(u) \\ G_\psi(h(x), M(x)u) &= M(x)G_\varphi(x, u) - \frac{1}{2}M'(x)(\rho_U(x)u)u. \end{aligned} \tag{2.3}$$

The first equation (2.3) is just (1.3) and the second provides the connection between the l.r. G_φ and G_ψ on overlaps. We have

Theorem 2.2 *A vector field S on E is a semispray if and only if it has l.r. S_φ in the form* (2.2) *and the functions involved in* (2.2) *satisfy* (2.3) *on overlaps of local charts.*

Proof. The "if" part was proved in the above. The converse is obvious. $\square$

We denote by $h_\lambda : E \to E$, $h_\lambda(u_x) = \lambda u_x$, $\lambda \in \mathbb{R}$, $\lambda > 0, x \in M$, the homothety of factor λ.

Definition 2.3 *We say that a semispray S is a spray if the following equality holds*

$$S \circ h_\lambda = \lambda(h_\lambda)_* \circ S. \tag{2.4}$$

Locally, (2.4) is equivalent to

$$G_\varphi(x, \lambda v) = \lambda^2 G_\varphi(x, v), \ \ (x, v) \in U \times \mathbb{E}. \tag{2.5}$$

Indeed, $(S \circ h_\lambda)(u) = S(\lambda u) = (x, \lambda v, \rho_U(\lambda v), -2G_\varphi(x, \lambda v)$ and $\lambda(h_\lambda)_* S(u) = (x, \lambda v, \lambda\rho_U(v), -2\lambda^2 G_\varphi(x, \lambda v))$. Since ρ_U is a linear mapping, (2.4) implies (2.5) and conversely. We look at (2.5). If we fix $x \in U$ and omit the index φ we get a mapping $G : \mathbb{E} \to \mathbb{E}$ that verifies $G(\lambda v) = \lambda^r G(v)$ for all $\lambda > 0$ and $r = 2$. We say that such a map is positively homogeneous of degree r.

For such mapping the following Euler type theorem holds.

Theorem 2.3 *Suppose that a mapping $G : \mathbb{E} \to \mathbb{E}$ is differentiable away from the origin of $\mathbb{E}$. Then the following two statements are equivalent:*

(i) G is positively homogeneous of degree r,

(ii) $dG_v(v) = rG(v)$, for all $v \in \mathbb{E}\backslash\{0\}$.

Proof. Suppose (i) holds. Fix $v \in \mathbb{E}$ and differentiate the equation $G(\lambda y) = \lambda^r G(v)$ with respect to the parameter λ. We get $dG_{\lambda v}(\lambda v) = r\lambda^{r-1}G(v)$ and for $\lambda = 1$, $dG_v(v) = rG(v)$, that is (ii) holds.

Conversely, suppose (ii), fix v and consider the mapping $\lambda \to G(\lambda v)$ with $\lambda > 0$. By the chain rule, we have $\dfrac{dG(\lambda v)}{d\lambda} = dG_{\lambda v}(v) = \dfrac{1}{\lambda} dG_{\lambda v}(\lambda v) = \dfrac{r}{\lambda} G(\lambda v)$, that is the mapping $\lambda \to G(\lambda v)$ is a solution of the differential equation $\frac{d}{d\lambda}G(\lambda v) - \frac{r}{\lambda}G(\lambda v) = 0$. The integrating factor $\frac{1}{\lambda^r}$ then gives $G(\lambda v) = \lambda^r C$, where the integrating constant C is depending on our fixed v. Setting $\lambda = 1$, we get $C = G(v)$ and so $G(\lambda v) = \lambda^r G(v)$, that is (i) holds, q.e.d. $\square$

The proof of Theorem 2.6 shows also that if $G : \mathbb{E} \to \mathbb{E}$ is of class C^1 on $\mathbb{E}$ and positively homogeneous of degree 1, then it is linear and $G(v) = dG_v(v)$. Moreover, if G is C^2 on $\mathbb{E}$ and is positively homogeneous of degree 2, then it is quadratic, that is $2G(v) = d_v^2 G(v, v)$.

Returning to the (2.5) we note that if G_φ is of class C^2 in the points $(x, 0)$, then it is quadratic in v. Thus S satisfying (2.4) reduces to a quadratic spray. In order to avoid this reduction we have to delete from E the image of the null section in the vector bundle $\pi : E \to M$.

Now, we show that if for a vector bundle $E \to M$ there exists a vector field S_0 on E that satisfies (2.4) then $\pi : E \to M$ is an anchored vector bundle and S_0 is a spray.

Let be $S_0(x, v) = (x, v, S_{01}(x, v), S_{0,2}(x, v))$ in a local chart on E. Then $S_0(h_\lambda u) = S_0(x, \lambda v) = (x, \lambda v, S_{01}(x, \lambda v), S_{02}(x, \lambda v))$ and $(h_\lambda)_* S_0(u) = (x, \lambda v, S_{01}(x, v), \lambda S_{02}(x, v))$. The condition (2.4) implies $S_{01}(x, \lambda v) = \lambda S_{01}(x, v)$ and

$S_{02}(x, \lambda v) = \lambda^2 S_{02}(x, v)$. It follows that S_{01} is a linear map with respect to v. Hence we may put $S_{01}(x, v) = \rho_U(x)v$, $\rho_U(x) \in L(\mathbb{E}, \mathbb{M})$. Using $\{\rho_U(x), x \in M\}$ one defines a morphism $\rho : E \to TM$. Thus $E \to M$ is an anchored vector bundle. As $(\pi_* \circ S_0)(u) = (x, S_{01}(x, v)) = (x, \rho_U(x)v)$ we have $\pi_* \circ S_0 = \rho$ and as $\tau_E \circ S_0$=indentity automatically holds it follows that S_0 is a spray.

3 Category of Banach Lie algebroids

Let $\pi : E \to M$ be an anchored Banach vector bundle with the anchor $\rho_E : E \to TM$ and the induced morphism $\rho_E : \Gamma(E) \to \mathcal{X}(M)$.

Assume there exists defined a bracket $[,]_E$ on the space $\Gamma(E)$ that provides a structure of real Lie algebra on $\Gamma(E)$.

Definition 3.1 *The triplet $(E, \rho_E, [,]_E)$ is called a Banach Lie algebroid if*

(i) $\rho : (\Gamma(E), [,]_E) \to (\mathcal{X}(M), [,])$ *is a Lie algebra homomorphism and*

(ii) $[s_1, fs_2]_E = f[s_1, s_2]_E + \rho_E(s_1)(f)s_2$, *for every* $f \in \mathcal{F}(M)$ *and* $s_1, s_2 \in \Gamma(E)$.

Example.

1. The tangent bundle $\tau : TM \to M$ is a Banach Lie algebroid with the anchor the identity map and the usual Lie bracket of vector fields on M.

2. For any submersion $\pi : E \to M$, the vertical bundle VE over E is an anchored Banach vector bundle. As the Lie bracket of two vertical vector fields is again a vertical vector field it follows that $(VE, i, [,]_{VE})$, where $i : VE \to TE$ is the inclusion map, is a Banach Lie algebroid. This applies, in particular, to any Banach vector bundle $\pi : E \to M$.

Let $\Omega^q(E) := \Gamma(\Lambda^q E^*)$ be the $\mathcal{F}(M)-$ module of differential forms of degree q. In particular, $\Omega^q(TM)$ will be denoted by $\Omega^q(M)$. The differential operator $d_E : \Omega^q(E) \to \Omega^{q+1}(E)$ is given by the formula

$$(d_E\omega)(s_0, \ldots, s_q) = \sum_{i=0,\ldots,q} (-1)^i \rho_E(s_i)\omega(s_0, \ldots, \widehat{s}_i, \ldots, s_q)$$

$$+ \sum_{0 \le i < j \le q} (-1)^{i+j} \omega([s_i, s_j]_E, s_0, \ldots \widehat{s}_i, \ldots, \widehat{s}_j, \ldots, s_q) \tag{3.1}$$

for $s_1, \ldots, s_q \in \Gamma(E)$, where hat over a symbol means that symbol must be deleted.

For Lie algebroids constructed on vector bundles with finite dimensional fibres there exist three different but equivalent notions of morphisms.

For Banach Lie algebroids only one of them is working. We give it here. For a detailed discussion on Lie algebroids morphisms see [4]. Let (E', π', M) be a Banach vector bundle and $(E', \rho_{E'}, [,]_{E'})$ a Banach Lie algebroids based on it.

Definition 3.2 *A vector bundle morphism $f : E \to E'$ over $f_0 : M \to M'$ is a morphism of the Banach Lie algebroids $(E, \rho_E, [,]_E)$ and $(E', \rho_{E'}, [,]_{E'})$ if*

the map induced on forms $f^* : \Omega^q(E') \to \Omega^q(E)$ *defined by* $(f^*\omega')_x(s_1, \ldots, s_q) = \omega'_{f_0(x)}(fs_1, \ldots, fs_q)$, $s_1, \ldots, s_2 \in \Gamma(E)$ *commutes with the differential i.e.*

$$d_E \circ f^* = f^* \circ d_{E'}. \tag{3.2}$$

Using this definition it is easy to prove

Theorem 3.1 *The Banach Lie algebroids with the morphisms defined in the above, form a category.*

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